VIBRATION ANALYSIS OF FLEXIBLE

CAM-FOLLOWER SYSTEMS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Bernard Auston Knight

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

March, 1965

VIBRATION ANALYSIS OF FLEXIBLE

CAM-FOLLOWER SYSTEMS

Approved:

Date Approved by Chairman: 4/5/65

ACKNOWLEDGMENTS

The author of this work would like to express his most sincere gratitude to all those persons who have contributed to the completion of this study. Dr. Harold L. Johnson, the faculty advisor, never fell short in providing encouragement and guidance. Appreciation is extended to Dr. A. H. Bailey and Dr. Takashi Nakada for their helpful comments in reviewing this work, especially to Dr. Nakada, for the many hours of help in the early stages of the study. The criticism and suggestions of Dr. F. R. E. Crossley at various stages of the work are also appreciated.

TABLE OF CONTENTS

Page

APPENDICES

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

Most analyses of flexible cam-follower systems to present, for practical reasons, involve the reduction of the system to a one degree of freedom approximation, which exhibits only one mode of vibration. In this study two methods of analyzing the more appropriate many degree of freedom representation of a system with a reasonable amount of time and effort are presented. The first is a numerical solution of the equations of motion of the many degree of freedom system using the Runge-Kutta method and the Burroughs 220 high speed digital computer, and the second method is a general formula for the response of any linear network forced by a functional input, derived by application of Duhamel's theory to the unit impulse response of the system.

The analysis of an overhead valve mechanism typical of internal combustion engines as a sample problem shows that the one degree of freedom system leaves something to be desired, especially with respect to velocity and acceleration responses.

Due to the nature of the analysis presented, the reaction force at the cam surface and the required precompression force in the valve spring to prevent jump are immediately available as a byproduct. This will allow the making of predictions previously not readily possible for design purposes. It is shown that the reaction force obtained by the operational calculus method of the four degree of freedom representation gives the first four terms in the Fourier cosine series

obtained by the continuum analysis, with the coefficients possibly modified to encourage more rapid convergence.

CHAPTER I

INTRODUCTION

The problem of designing a mechanism to produce a given motion, velocity, and acceleration very often occurs in machine design. A popular and expedient solution is the use of miscellaneous contour surfaces called cams. More specifically, a cam may be thought of as a function generator, whose input is a constant velocity, either rotative or linear, and whose output is a particular function of time. Cams of some form or other are found in many machines; for example internal combustion engines, shoe making machinery, textile machinery, ejection molds, and computers to name a few.

Statement of the Problem

Output Response of the System

Once a mechanism and its driving cam have been designed, the question arises as to how well the output link of the mechanism follows the command of the cam. For most systems the response can be considered exact at static positions or low speeds, provided the static deflection of the links is included. But, what about the response at high cam speeds? Any physical system is indeed somewhat flexible and has mass distributed through it. Both mass and flexibility affect the dynamic response of a mechanical system.

At this point it is appropriate to introduce a particular cam-follower mechanism and use it as the basis of all further development in this work, bearing in mind, however, that the principles herein presented are intended to be sufficiently general that they can be applied to any mechanical system. Consider the sample system to be an overhead poppet valve linkage such as that found in many internal combustion engines. See Figure 1. The overhead valve linkage is so popular, in fact, that several dozen papers have been written on its design and dynamic response. They may be separated into three distinct areas of investigation:

1. Quantitative prediction of high speed valve motion for a given cam and linkage.

2. Design of cams to produce a particular motion at a particular cam speed.

3. Qualitative investigations of the general principles of the dynamics of cam-actuated systems.

In the first area there are papers by Olmstead and Taylor,[1] and Barkan.[5] Barkan's paper is based on his Doctor of Philosophy dissertation,[6] which is by far the most comprehensive work known to the author on the topic. In the second area are papers by Dudley[7] and Stoddart,[8] in which the problem becomes a synthesis rather than an analysis. The major limitation of this method is that a cam designed to work well at one speed will produce serious linkage vibrations and consequently deviation of the follower response from that desired at other speeds. Lastly, in the third area of investigation are papers by Hrones,[9] Mitchell,[11] and Warming.[12] The papers of Hrones and Warming show that sudden changes in the acceleration of the cam profile and proper phasing of the acceleration forces produce large vibrations in the output link, which may even grow unbounded. This phenomenon is also discussed in some detail by

Jacobsen and Ayre.[13] The work by Mitchell confirmed experimentally the claim by Hrones that sudden changes in the input acceleration induces vibratory amplitudes of output acceleration twice as large as the input value.

There is one point in common with every single work mentioned above. It is that the overhead valve linkage shown in Figure 1 has been reduced to a single degree of freedom system (one point mass and one spring) as shown in Figure 2. This is a rather gross approximation, but very pragmatic in that the analysis is simple. The point mass is simply the total dynamic mass of the system, and the spring is chosen so that the natural frequency of the one degree of freedom system coincides with the first mode natural frequency of the many degree of freedom system representing the actual linkage. Such a method is a good engineering approximation in that it gives good results some of the time as will be shown in Chapter VII. According to Rothbart[14]:

> However, in the range of resonance values [vibrational amplitudes] may have magnitudes requiring investigation when the cam speed agrees with the natural vibratory mode. Tests of the natural frequencies of the frame may be necessary. More exact methods may require the basis of two or more degrees of freedom -- generally, calculating time and effort may be prohibitive.

Reaction Force at the Cam

Another problem area associated with flexible cam-follower systems is the reaction force at the cam. The literature is practically devoid of work in this area. There is only the comment by Talbourdet* in the discussion of the paper by Hrones[9]:

---

*Mechanical Engineer, Research Division, United Shoe Machinery Corporation, Beverly, Mass., Member ASME.

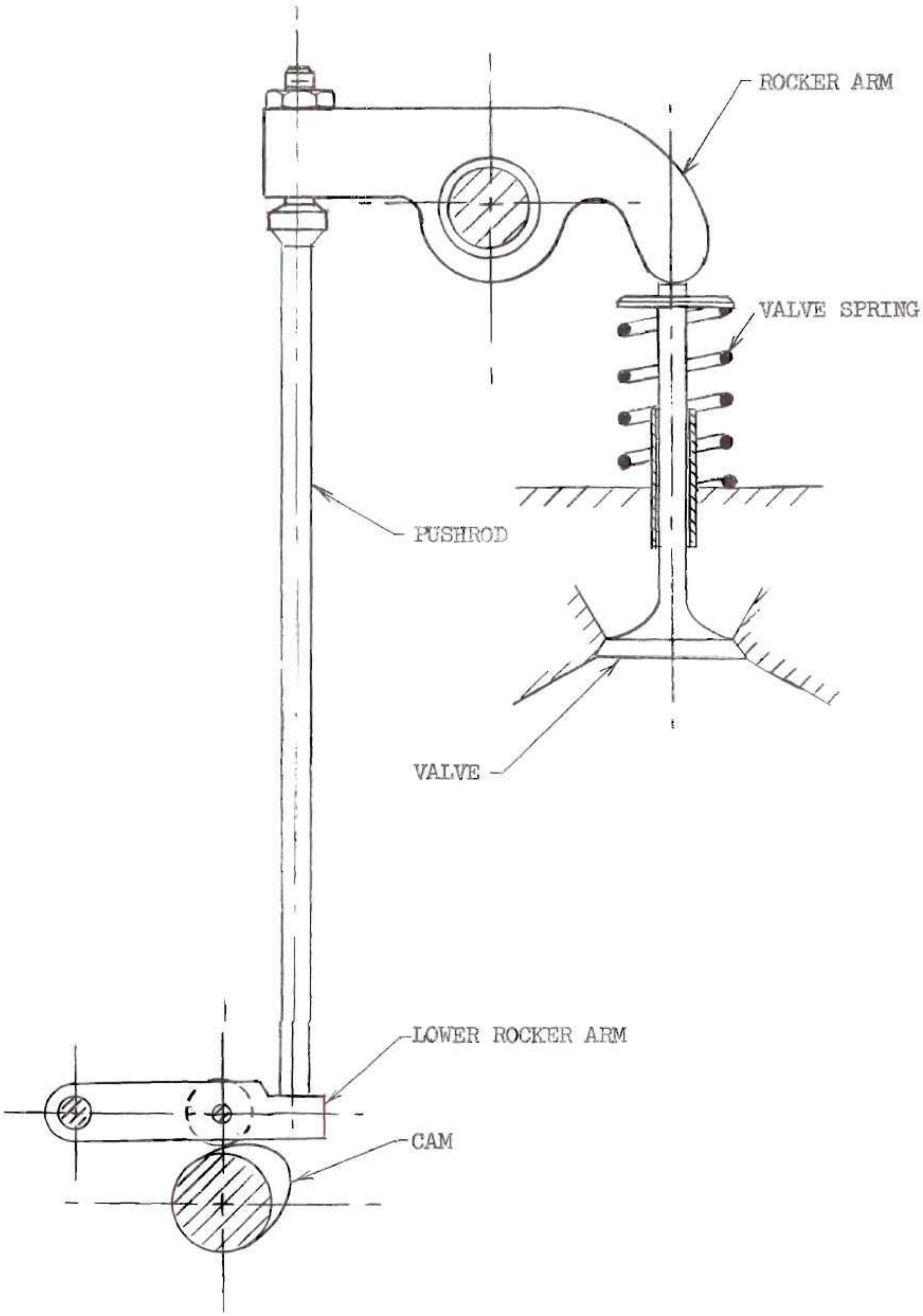


Figure 1. Typical Overhead Valve Linkage.

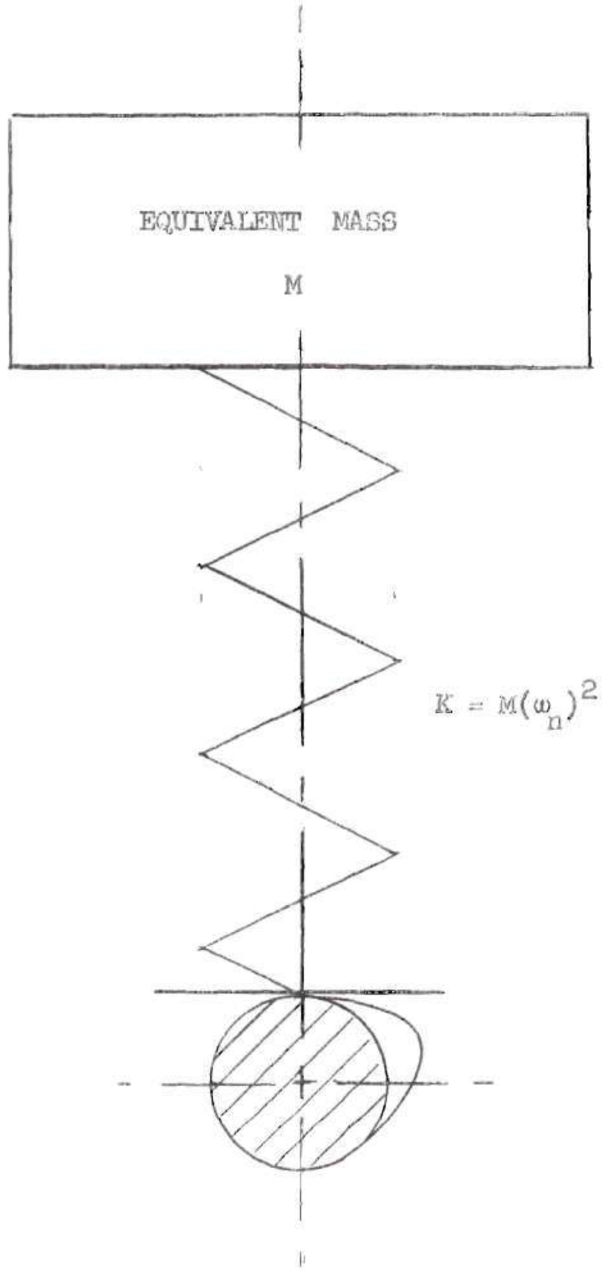


Figure 2. One Degree of Freedom Approximation of Overhead Valve Linkage.

> For many years we have been engaged in determining from tests the surface-endurance limits of materials for cams to enable us to calculate the safe load-carrying capacity of cam surfaces, and it may be mentioned that the results of these tests have been invaluable in cam design. However, we are in the dark regarding the extent of the dynamic forces imposed on the surfaces of cams imparting to followers some well-known motions such as gravity, harmonic, and cycloidal, or any combination of these motions, although in some few cases we have been able, from the extent of the surface failure of a cam track and the number of repeated stresses, to obtain some indication of the critical dynamic loads imposed on the surface and to select a cam material to withstand these loads.

and an article by Rothbart[14] treating the topic in a cursory manner.

Methods of Analysis

It is the purpose of this study to present two methods requiring only a reasonable amount of time and effort to analyze the dynamic response and reaction forces at the cam due to a functional input by treating the system as a several degree of freedom system. By such an analysis, several modes of vibration will appear in the output, and resonance with any one of them can be spotted. The first method is a numerical solution of the differential equations of motion based on the well-known Runge-Kutta method, using a high speed digital computer. Even though computers are becoming more and more readily available, still they are not accessible to everyone. Therefore, the second method is a completely analytical solution requiring only pencil and paper. However, a good desk calculator would be useful. The analytic solution is a general formula for determining the response of a linear network to a functional input, derived by application of Duhamel's theory to the impulse response of the system in operational form. There is essentially nothing new in this use of Heaviside's calculus; it is the application of automatic control techniques to a dynamics problem.

Before treating the example of the overhead valve linkage it is necessary to develop the theory of the methods to be used, which is the purpose of the next two chapters. In Chapter IV the discussion of the overhead valve linkage will be resumed.

CHAPTER II

THEORY OF THE COMPUTER SOLUTION

By Newton's methods it is quite simple to write the equations of motion for a many-degree-of-freedom system, as will be shown in a later chapter. The most desirable next step is to obtain the solutions for these equations resulting from known input functions. These solutions can be obtained in a very straight forward manner by use of the well known Runge-Kutta method in conjunction with the high speed digital computer.

Basically the problem is to calculate additional points (x_1, y_1) of the integral curve, given $y' = f(x,y)$ and (x_o, y_o). Let $\Delta x = h$ and $\Delta y = k$ so that

$$y_1 = y_o + k, \quad \text{and} \quad x_1 = x_o + h \tag{1}$$

The problem is now reduced to finding values of k as refined as is necessary for decent accuracy. There have been several methods presented for finding k based on the Taylor series expansion

$$k = y_o{}'h + y_o{}''\frac{h^2}{2!} + y_o{}'''\frac{h^3}{3!} + y_o^{(4)}\frac{h^4}{4!} + \dots . \tag{2}$$

By far the easiest to calculate are the formulas of Runge, which have been modified by Heun and Kutta. The chief advantage of these formulas is that they can be written with the functional values only, thereby avoiding calculation of tedious derivatives. Let $k_{1,2,3,\dots}$ be the successive approximations of **k**, which will be properly weighted and averaged as indicated by equation (3) which follows. The formulas as modified by Kutta[4] are

$$
\begin{aligned}
k_1 &= f(x_o, y_o)\, h \\
k_2 &= f(x_o + \tfrac{h}{2}, y_o + \tfrac{k_1}{2})\, h \\
k_3 &= f(x_o + \tfrac{h}{2}, y_o + \tfrac{k_2}{2})\, h \\
k_4 &= f(x_o + h, y_o + k_3)\, h
\end{aligned}
\tag{a}
$$

and

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\,. \tag{3}$$

If y does not appear on the right hand side of the differential equation, the problem reduces to a simple quadrature and Kutta's equations take the same form as Simpson's rule. Therefore, the method may be thought of as approximating segments of the integral curve by parabolic arcs instead of straight line segments as in the Euler and modified Euler methods.

The method can be extended to systems of simultaneous differential equations and to equations of higher order than one. Suppose the differential equation has the form

$$y^{(n)} = f(x, y', y'', \ldots, y^{(n-1)})\,. \tag{4}$$

Equation (4) can be transformed into a system of n simultaneous first order differential equations by introducing a new variable for each derivative, as will be illustrated later. First consider, for example, two simultaneous equations

$$y' = f(x,y,z) \tag{5}$$

and

$$z' = g(x,y,z) \,. \tag{6}$$

By defining q such that

$$z_1 = z_o + q \tag{b}$$

k and q can be written similarly to equation (3) as follows:

$$\begin{aligned}
k_1 &= f(x_o \quad , y_o \quad , z_o \quad) \, h \\
q_1 &= g(x_o \quad , y_o \quad , z_o \quad) \, h \\
k_2 &= f(x_o + \tfrac{h}{2} , y_o + \tfrac{k_1}{2} , z_o + \tfrac{q_1}{2}) \, h \\
q_2 &= g(x_o + \tfrac{h}{2} , y_o + \tfrac{k_1}{2} , z_o + \tfrac{q_1}{2}) \, h \\
k_3 &= f(x_o + \tfrac{h}{2} , y_o + \tfrac{k_2}{2} , z_o + \tfrac{q_2}{2}) \, h \\
q_3 &= g(x_o + \tfrac{h}{2} , y_o + \tfrac{k_2}{2} , z_o + \tfrac{q_2}{2}) \, h \\
k_4 &= f(x_o + h , y_o + k_3 , z_o + q_3) \, h \\
q_4 &= g(x_o + h , y_o + k_3 , z_o + q_3) \, h
\end{aligned} \tag{c}$$

and

$$\begin{aligned}
k &= \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\
q &= \frac{1}{6} (q_1 + 2q_2 + 2q_3 + q_4)
\end{aligned} \tag{7}$$

It can be shown that the error in these approximations is proportional to h to the fifth power.[4] However, in order to check the accuracy, the difference of the factors of h^5 are not calculated. That would be extremely time-consuming. It is much easier to repeat the calculations for a double interval, $2h$, and compare the results. For the fourth order approximation used in this work, the error after a double increment of $2h$ is about 1/15 of the difference between the two results.[4] However, the method of securing accuracy in this work was a little less sophisticated but very simple to perform. Simply rerun the computer program at half the increment h for a short interval of x. The number of significant figures found in agreement were considered to be an indication of the accuracy of the results.

The computer program for the Burroughs 220 digital computer was designed to be as standardized as possible; that is to say, new problems can be worked by changing a minimum number of cards. Several examples of the program used for various problems in this work are given in Appendix B. An example of how to set a problem into the computer language follows:

Consider the differential equation to be

$$D^2x + 5Dx + 7x = 0 \ , \qquad (8)$$

where $D = d/dt$, having initial conditions $x(0) = 0$ and $Dx(0) = 10\,\frac{\sqrt{3}}{2}$. The solution is known to be

$$x(t) = 10\,e^{-\frac{5}{2}t} \sin \frac{\sqrt{3}}{2}\,t \ . \qquad (9)$$

Let the upper case characters that follow be terms written in Algol[10] (computer language) using the Hollerith character set. Define

$$X(1) = x$$

and

$$X(2) = \frac{dx}{dt} = x'$$

so that

$$\frac{d}{dt} X(1) = X(2)$$

and

$$\frac{d}{dt} X(2) = x'' = -5(X(2)) - 7(X(1))$$

becomes a system of first order equations representing equation (8). The program with a sample of its printout for the example above is given beginning on page 66 of Appendix B. In the program D(1,K) and D(2,K) correspond to k and q in the previous discussion. Further, the exact solution given by equation (9) has been written into the program as X1C and printed out in the column next to X1 for purposes for comparison.

CHAPTER III

DEVELOPMENT OF THE CLOSED FORM ANALYSIS

The output of any linear network as shown in Figure 3, which is forced by a functional input, can be determined by applying Duhamel's theory in connection with the known impulse response of the system. See the schematic in Figure 3B.

By the Heaviside's series expansion of the operational transfer function $G(s)$ of the "black box" shown in Figure 3B, the impulse response can be calculated as follows[3]:

$$W(t) = \sum_{k=1}^{r} \frac{P(s_k)}{Q'(s_k)} e^{s_k t}, \tag{10}$$

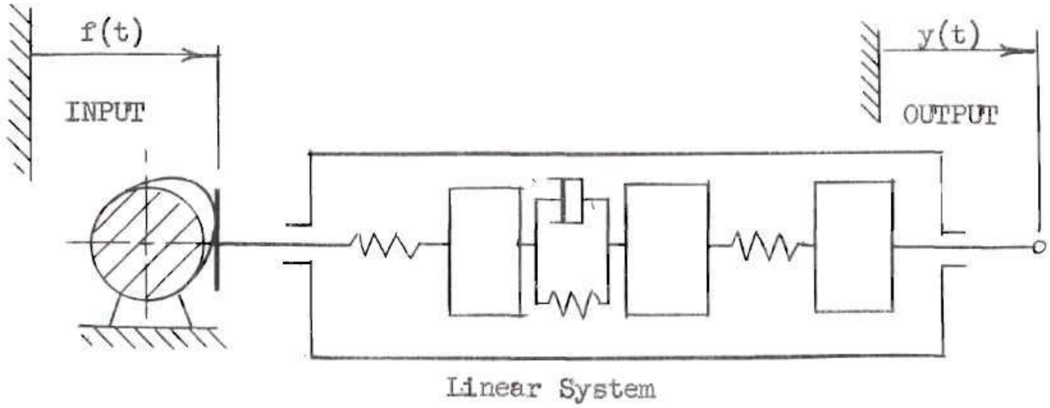
where s_k are the successive roots of $Q(s) = 0$. Some roots are real numbers, but other roots are pairs of complex conjugates, $(a_k \pm jb_k)$.

The response of the system resulting from the application of any input signal $f(t)$ can be formulated by Duhamel's theorem[2] as being

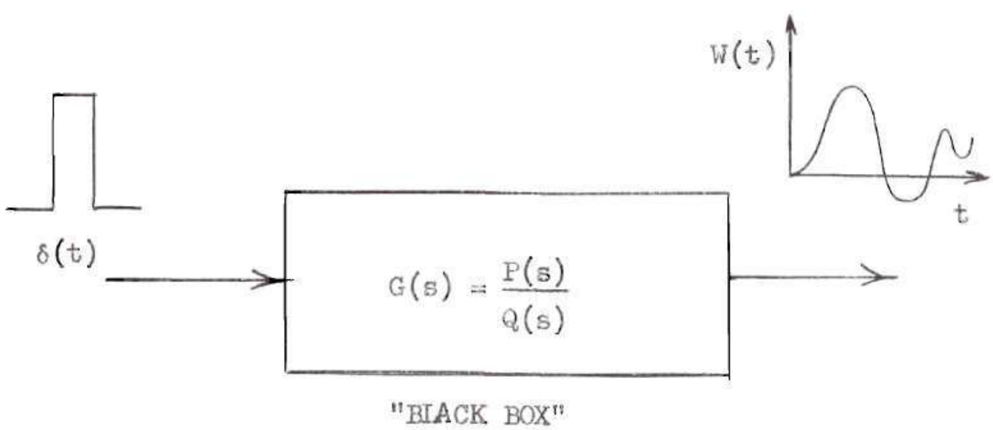
$$y(t) = \int_0^t f(t)\, W(t-\tau) d\tau . \tag{11}$$

The substitution of equation (10) into equation (11) gives the following formula:

$$y(t) = \int_0^t \sum_{k=1}^{r} \frac{P(s_k)}{Q'(s_k)} s^{s_k(t-\tau)} f(\tau) d\tau . \tag{12}$$



(A)



(B)

Figure 3. Linear Network Under Input Signal $f(t)$ or $\delta(t)$.

The essential part of the integration by τ is

$$\int_o^t e^{-s_k \tau} f(\tau)\, d\tau ,$$

which can be evaluated by repeatedly integrating by parts as follows:

$$\int_o^t e^{-s_k \tau} f(\tau)\, d\tau = \left[\frac{e^{-s_k \tau} f(\tau)}{-s_k} \right]_o^t - \int_o^t \frac{e^{-s_k \tau} f'(\tau)}{-s_k}\, d\tau \qquad \text{(a)}$$

$$= \left(\frac{f(0) - f(t)e^{-s_k t}}{s_k} \right) + \frac{1}{s_k} \int_o^t e^{-s_k \tau} f'(\tau) d\tau ,$$

where

$$f'(\tau) = \frac{d}{d\tau} \left\{ f(\tau) \right\} .$$

Noting that the integral of the right hand side of equation (a) is similar to the left hand side of equation (a), the equation can be expanded into a recursion formula as follows:

$$\int_o^t e^{-s_k t} f(\tau) d\tau = \sum_{n=o}^{n} \frac{f^{(n)}(0) - f^{(n)}(t) e^{-s_k t}}{s_k^{\,n+1}} \qquad (13)$$

$$+ \frac{1}{s_k^{\,n+1}} \int_o^t e^{-s_k \tau} f^{(n+1)}(\tau) d\tau .$$

In equation (13) the integral of the $n+1^{th}$ derivative of $f(\tau)$ is to be evaluated. However, if the input signal is a polynomial form in t, as is the case for very many cam designs, and is expressed by a finite number of terms, $f^{(n+1)}(\tau)$ will be zero. Now equation (12) can be calculated by means of equation (13) as follows:

$$y(t) = \int_o^t \sum_{k=1}^{r} \frac{P(s_k)}{Q'(s_k)} e^{s_k(t-\tau)} f(\tau)\, d\tau$$

$$= \sum_{k=1}^{r} \frac{P(s_k)}{Q'(s_k)} e^{s_k t} \int_o^t e^{-s_k \tau} f(\tau) d\tau$$

$$= \sum_{k=1}^{r} \frac{P(s_k)}{Q'(s_k)} \left[\left(\sum_{n=o}^{n} \frac{f^{(n)}(0)\, e^{s_k t} - f^{(n)}(t)}{s_k^{\,n+1}} \right) + \frac{e^{s_k t}}{s_k^{\,n+1}} \int_o^t e^{-s_k \tau} f^{(n+1)}(\tau)\, d\tau \right].$$

By grouping similar terms,

$$y(t) = \sum_{k=1}^{r} \sum_{n=o}^{n} \frac{P(s_k)}{Q'(s_k)} \cdot \frac{e^{s_k t}}{s_k^{\,n+1}} f^{(n)}(0) - \sum_{k=1}^{r} \sum_{n=o}^{n} \frac{P(s_k)}{Q'(s_k)} \cdot \frac{f^{(n)}(t)}{s_k^{\,n+1}}$$

$$+ \sum_{k=1}^{r} \frac{P(s_k)}{Q'(s_k)} \cdot \frac{1}{s_k^{\,n+1}} \int_o^t e^{s_k(t-\tau)} f^{(n+1)}(\tau)\, d\tau . \qquad (14)$$

Equation (14) is the general formula, including everything, for the problem at hand. When there is damping in the system the roots of $Q(s_k)$ have negative real parts. As a result the first term of the right hand side of the equation (14) will be damped out in the course of time, and thus it can be considered to be the influence of the initial condition of the input $f(t)$.

In the practical case, some roots of $Q(s_k) = 0$ will be real numbers, but others may indeed be pairs of complex conjugates of the form

$$\left.\begin{aligned} s_k &= a_k + jb_k \\ \bar{s}_k &= a_k - jb_k \end{aligned}\right. \quad , \quad p < k < r \; .$$

If the value of $P(s_k)/Q'(s_k)$ is represented by the substitution of $s_k = a_k + jb_k$, it will have the form $A_k + jB_k$ after all real parts and all imaginary parts are collected. Further, when $P(s_k)/Q'(s_k)$ is analytic, $P(\bar{s}_k)/Q'(\bar{s}_k)$ will have the form $A_k - jB_k$. These complex numbers will be reformed as follows:

$$s_k = a_k + jb_k = \sqrt{a_k^2 + b_k^2} \cdot e^{j\theta_k}$$

$$\bar{s}_k = a_k - jb_k = \sqrt{a_k^2 + b_k^2} \cdot e^{-j\theta_k} \; ,$$

where

$$\tan \; \theta_k = \frac{b_k}{a_k} \; .$$

Also

$$\frac{P(a_k \pm jb_k)}{Q'(a_k \pm jb_k)} = A_k \pm jB_k = \sqrt{A_k^2 + B_k^2} \; e^{\pm j\varphi_k}$$

$$= \left| \frac{P(s_k)}{Q'(s_k)} \right| , \; e^{\pm j\varphi_k} \; ,$$

where

$$\tan \; \varphi_k = \frac{B_k}{A_k} \; .$$

For the summation of a pair of conjugate roots, s_k and $\bar{s}_k$, in the first term of the right hand side of equation (14)

$$\frac{P(s_k)}{Q'(s_k)} \cdot \frac{e^{s_k t}}{s_k^{\,n+1}} + \frac{P(\bar{s}_k)}{Q'(\bar{s}_k)} \cdot \frac{e^{\bar{s}_k t}}{\bar{s}_k^{\,n+1}} \tag{b}$$

$$= \frac{\sqrt{A_k^2 + B_k^2}}{\left(\sqrt{a_k^2 + b_k^2}\right)^{n+1}} \; e^{a_k t} \left[e^{j\left\{b_k t + \varphi_k - (n+1)\theta_k\right\}} + e^{-j\left\{b_k t + \varphi_k - (n+1)\theta_k\right\}} \right]$$

$$= 2 \left| \frac{P(s_k)}{Q'(s_k)} \right| \cdot \frac{e^{n_k t}}{|s_k|^{n+1}} \cdot \cos\left\{ b_k t + \varphi_k - (n+1)\theta_k \right\}.$$

Similarly, the summation of a pair of conjugate roots, s_k and $\bar{s}_k$ in the second term in the right hand side of equation (14) becomes

$$\frac{P(s_k)}{Q'(s_k)} \cdot \frac{1}{s_k^{\,n+1}} + \frac{P(\bar{s}_k)}{Q'(\bar{s}_k)} \cdot \frac{1}{\bar{s}_k^{\,n+1}} \tag{c}$$

$$= 2 \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{1}{|s_k|^{n+1}} \cdot \cos\left\{ \varphi_k - (n+1)\theta_k \right\}.$$

Also the last term in equation (14) becomes

$$\frac{P(s_k)}{Q'(s_k)} \frac{1}{s_k^{\,n+1}} \int_0^t e^{s_k(t-\tau)} f^{(n+1)}(\tau)d\tau + \frac{P(\bar{s}_k)}{Q'(\bar{s}_k)} \frac{1}{\bar{s}_k^{\,n+1}} \int_0^t e^{\bar{s}_k(t-\tau)} f^{(n+1)}(\tau)d\tau$$

$$= 2 \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{1}{|s_k|^{n+1}} \int_0^t e^{a_k(t-\tau)} \cdot \cos\left\{ b_k(t-\tau) + \varphi_k - (n+1)\theta_k \right\} f^{(n+1)}(\tau)d\tau \tag{d}$$

By substituting equations (b), (c), and (d) into equation (14) the final result becomes

$$y(t) = 2 \sum_{k=p}^{r} \sum_{n=o}^{n} \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{e^{a_k t}}{|s_k|^{n+1}} \cdot \cos\left\{b_k t + \varphi_k - (n+1)\theta_k\right\} f^{(n)}(0)$$

$$- 2 \sum_{k=p}^{r} \sum_{n=o}^{n} \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{1}{|s_k|^{n+1}} \cdot \cos\left\{\varphi_k - (n+1)\theta_k\right\} f^{(n)}(t)$$

$$+ 2 \sum_{k=p}^{r} \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{1}{|s_k|^{n+1}} \int_o^t e^{-a_k(t-\tau)} \cos\left\{b_k(t-\tau) + \varphi_k - (n+1)\theta_k\right\}$$

$$\cdot f^{(n+1)}(\tau)\, d\tau \qquad (15)$$

for complex roots of $Q(s_k) = 0$, where for purposes of summary

$$G(s) = P(s)/Q(s) \; ,$$

and

$$Q'(s) = \frac{d}{dt}\left\{Q(s)\right\} .$$

Also $k = 1$ to m for s_k real, and $k = p$ to r for s_k and $\bar{s}_k$ being conjugate pairs of complex roots. It should be remembered that for s_k real, the general formula for the problem remains equation (14). However, this is rarely the case in real mechanical systems. That is to say, s_k is complex in most real situations.

CHAPTER IV

ANALYSIS OF THE ONE DEGREE OF FREEDOM SYSTEM

Before delving into the heart of the purpose of this study -- analysis of a many degree of freedom system -- it is necessary to show how the many degree of freedom system and the one degree of freedom approximation are derived from the actual mechanism and to demonstrate the analysis of the one degree of freedom approximation. The one degree of freedom approximation will be analyzed by the classical method, the operational calculus method, and the computer method. The forcing function will be considered to be a parabolic cam curve of the form

$$Z = A_o + A_1 t + A_2 t^2 \ . \tag{16}$$

See Appendix A. There is good reason for using the parabolic cam curve in the analysis, even though it is not the best cam curve in actual practice due to the step changes in input acceleration. The parabolic cam curve is the simplest polynomial input, for which the operational calculus solution is suitable, that retains all the mathematical characteristics of higher order polynomials. The following demonstrations, therefore, will be as simple as possible and yet complete.

Development of the One Degree of Freedom System From the Actual Mechanism

In order to set the problem up for mathematical treatment, the mechanism must be converted into a lumped-parameter system. The evolution of such a system is depicted in Figures 4, 5, 6, and 7. The methods, as described by Barkan,[5] are fairly conventional, and consequently only a sketch and explanation of the steps is offered here in lieu of a full and detailed description.

Each part itself is converted into a lumped system. The reciprocating parts are weighed and their stiffness values are measured experimentally. Then each reciprocating part becomes a massless spring with a point mass at each end. The dotted regions on system A in Figure 4 show how the system is lumped. The rotating parts (upper and lower rockers) are converted into point masses on the ends of massless flexible rods having the same moment of inertia I as do the real parts. Figure 5 and Table 1 shows the results of this conversion of the real system into a lumped-parameter system and typical values for the masses and stiffness values, as used by Barkan.[6]

The next step is to develop a system which is dynamically equivalent to the intermediate system B, but having all parts acting in the valve datum, since the motion of the valve is the item of interest. The cam is replaced with an equivalent cam having lift $R.Z_o = Z$, where R is the lever ratio between the cam and the valve,

$$R = \frac{\text{valve lift}}{\text{cam lift}} .$$

R is derived from the geometry of the linkages. The resulting system C is shown in Figure 6 with the relationship between the masses and the stiffness values in the intermediate system B. The evolution of the intermediate system C is further based on the requirement that potential and kinetic energies of the corresponding components be equal. It can be shown[6] that this requirement is satisfied when

$$M' = \frac{M''}{(r_{ij})^2} \qquad (17)$$

and

$$K' = \frac{K''}{(r_{ij})^2}\ , \qquad (18)$$

where

$$r_{ij} = \frac{\text{lift at datum i}}{\text{lift at datum j}}\ .$$

Also shown in Figure 6 is the valve spring, denoted by K_s.

The one degree of freedom approximation follows readily from system C as is shown in Figure 7. The mass is the total dynamic mass of system C,

$$M = M_1 + M_2 + M_3 + M_4\ , \qquad (19)$$

and the equivalent linkage stiffness, K_L, is designed so that the one degree of freedom system has the same natural frequency as the first mode frequency, ω_n, of system C. The normal way to find this first mode frequency is by a Holzer tabulation or any other well known method. In this work the many degree of freedom system C was analyzed first. Therefore ω_n was already known from the period $\lambda = 0.002$ seconds.

$$\omega_n = \frac{2\pi}{\lambda} = \frac{2\pi}{0.002} = 2131.49 \text{ rad/sec}$$

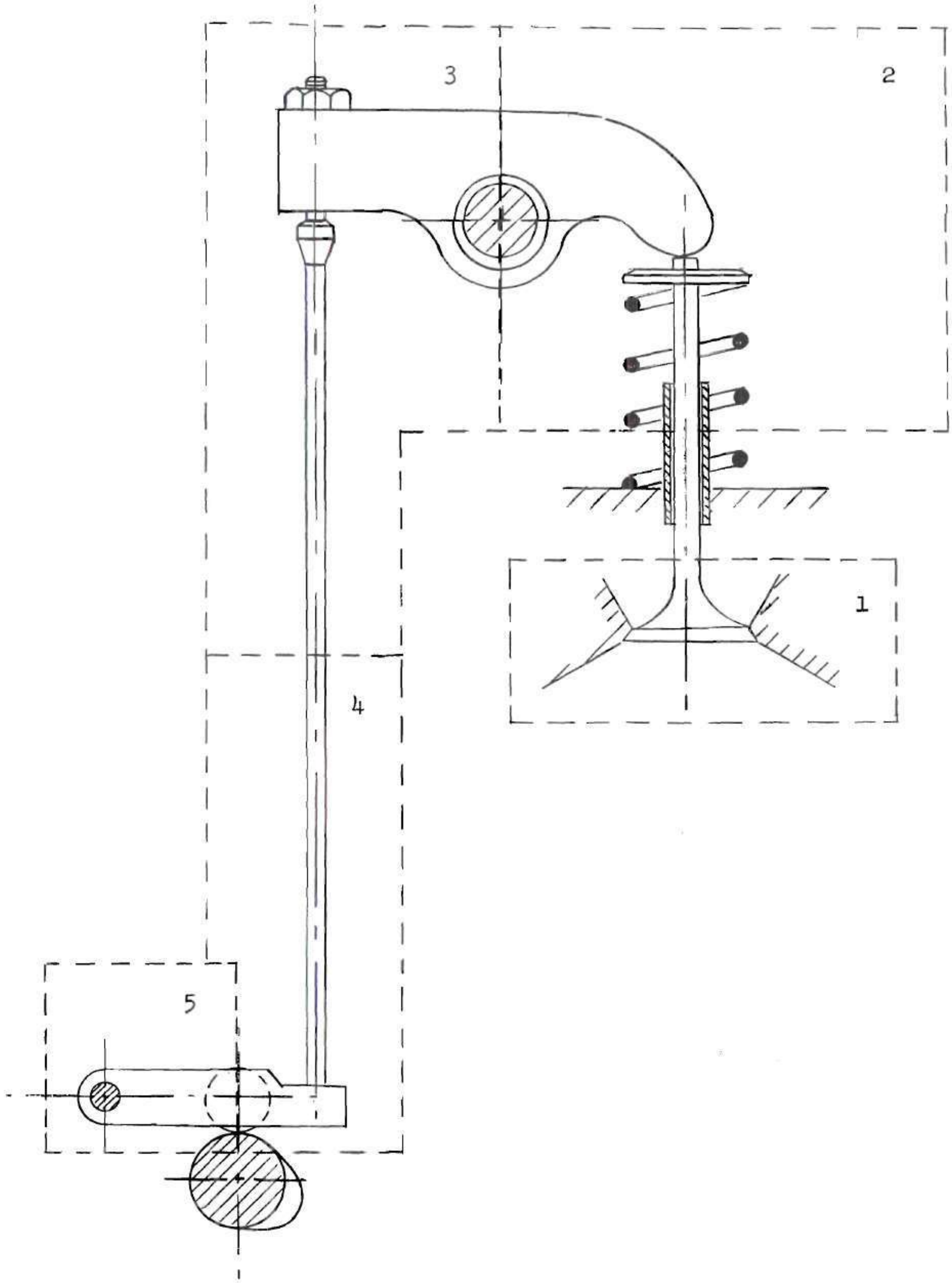


Figure 4. Linkage of the Sample Problem, System A.

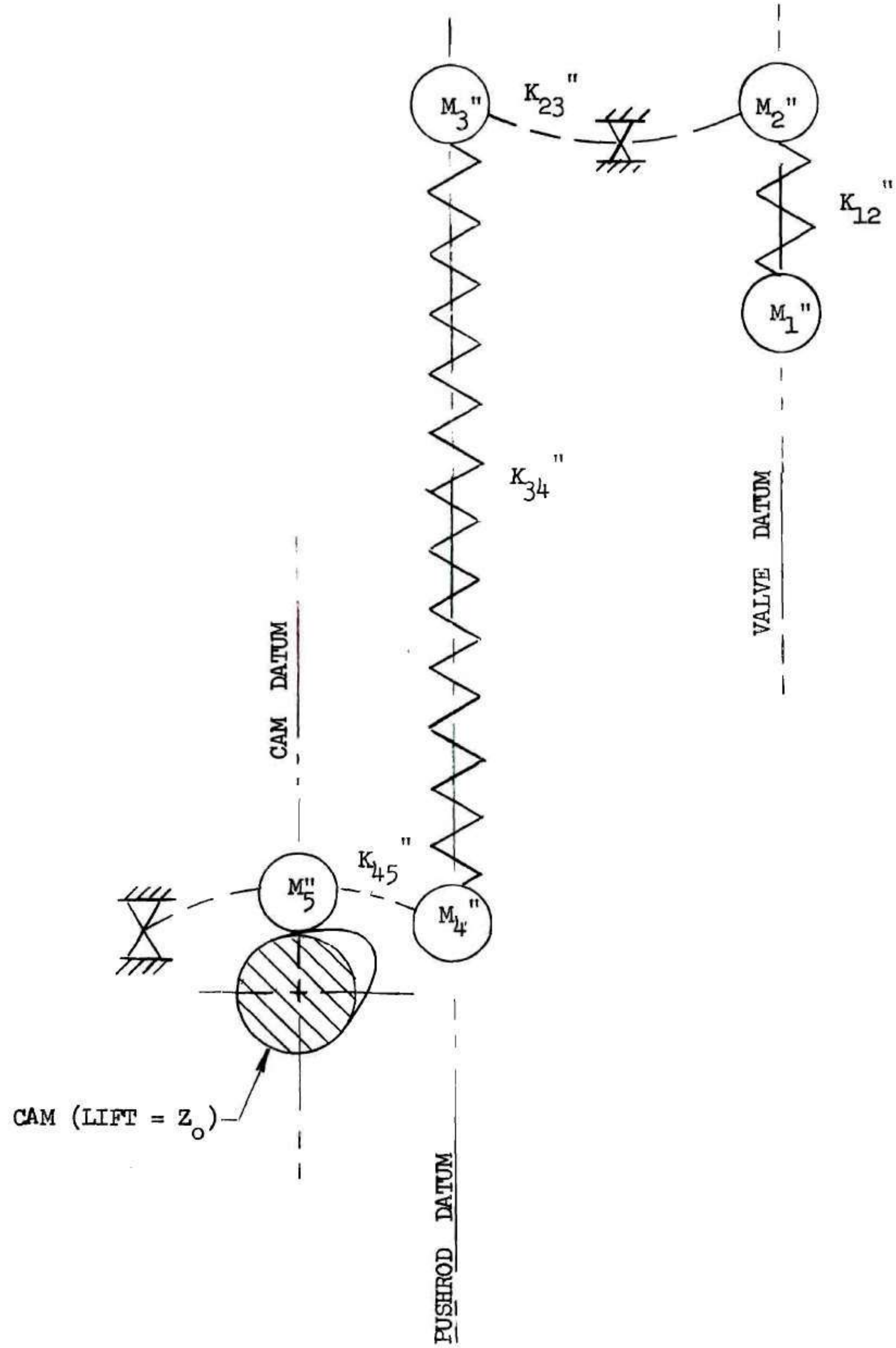


Figure 5. Intermediate System B.

Table 1. Component Masses and Stiffness Values for Intermediate System B.

M_1"(valve head + $\frac{1}{2}$ valve stem)	$= 1.333 \times 10^{-3}\ \frac{\text{lb-sec}^2}{\text{in}}$
M_2" ($\frac{1}{2}$ valve stem + spring cap and keepers + $\frac{1}{3}$ valve spring* + upper rocker arm (valve side))	$= 1.075 \times 10^{-3}\ \frac{\text{lb-sec}^2}{\text{in}}$
M_3" (upper rocker arm (pushrod side) + $\frac{1}{2}$ pushrod)	$= 0.972 \times 10^{-3}\ \frac{\text{lb-sec}^2}{\text{in}}$
M_4" ($\frac{1}{2}$ pushrod + lower rocker lever (pushrod section))	$= 0.785 \times 10^{-3}\ \frac{\text{lb-sec}^2}{\text{in}}$
M_5" (lower rocker lever (cam section))	$= 1.223 \times 10^{-3}\ \frac{\text{lb-sec}^2}{\text{in}}$
K_{12}"(valve stem)	= 300,000 lb/in
K_{23}"(upper rocker arm)	= 71,500 lb/in
K_{34}"(pushrod)	= 137,500 lb/in
K_{45}"(lower rocker lever)	= 236,500 lb/in

*See Appendix C.

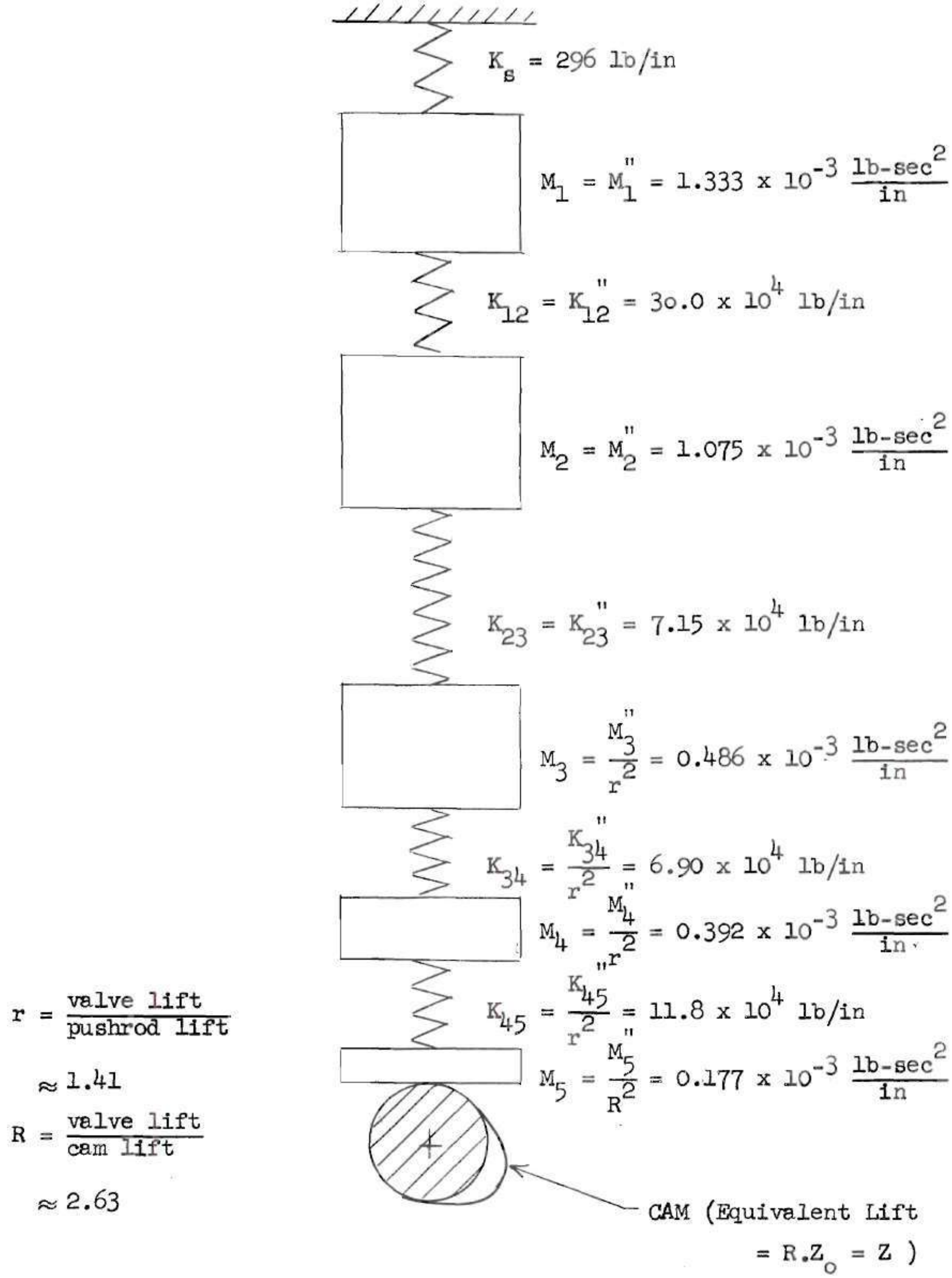


Figure 6. Intermediate Equivalent System C.

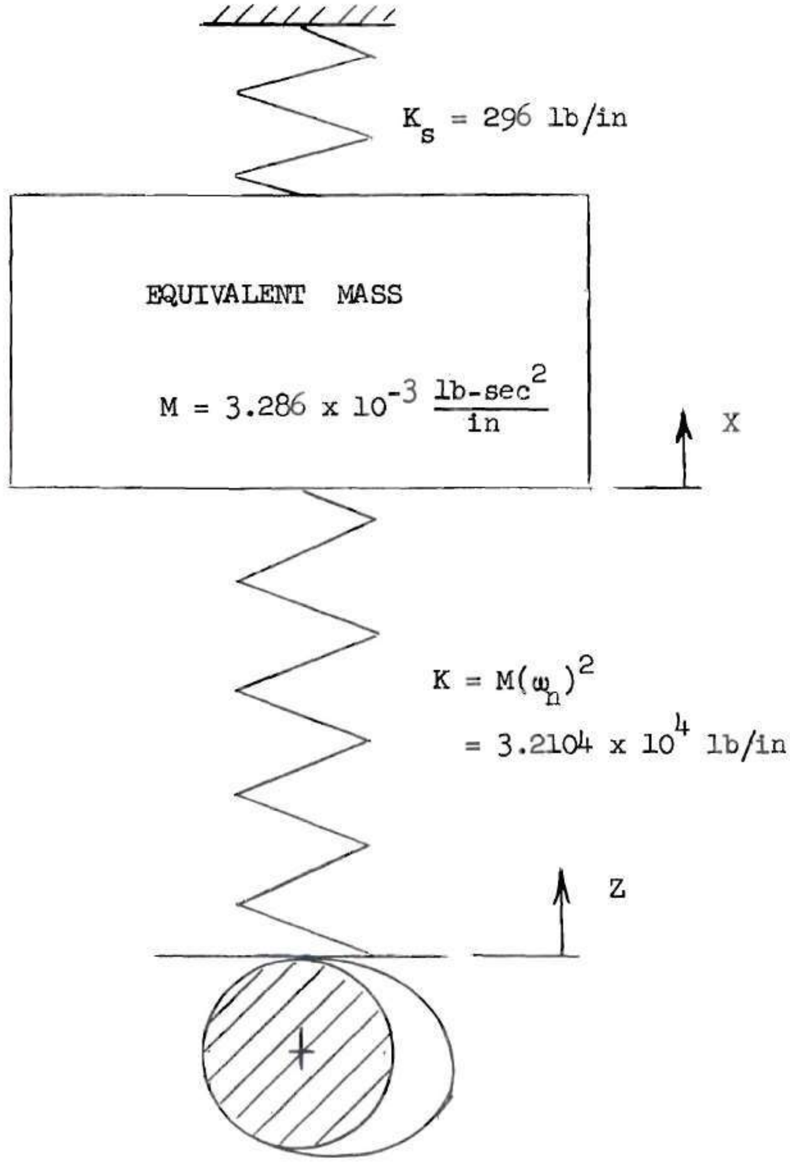


Figure 7. One Degree of Freedom Approximation, System D.

$$K = M(\omega_n)^2 = (3.286 \times 10^{-3})(9.85 \times 10^6)$$

$$= 3.24 \times 10^4 \text{ lb/in}$$

$$K = K_s + K_L$$

$$K_L = 3.24 \times 10^4 - K_s$$

$$= 3.24 \times 10^4 - 296 = 3.2104 \times 10^4 \text{ lb/in}$$

Classical Analysis with Parabolic Cam Curve

Since the one degree of freedom system can be analyzed by classical methods, this will be done to provide an exact comparison with the results of the operational calculus method.

The equation of motion of the system D shown in Figure 7 is

$$D^2x + \frac{K}{M}x = \frac{K_L}{M}Z \tag{20}$$

where $K = K_s + K_L$, $D^2x = d^2x/dt^2$, and $Z = \frac{H}{2\beta^2}t^2$, $0 \leqq t \leq \beta$,* for the first half of the rise. Therefore equation (20) becomes

$$D^2x + p^2x = At^2 \tag{21}$$

where $p^2 = K/M$ and $A = K_L H/2\beta^2 M$. The initial conditions are $x = 0$ and $dx/dt = 0$ at $t = 0$. The general solution of equation (21) is

$$x(t) = C_1 \sin(pt) + C_2 \cos(pt) + \frac{At^2}{p^2} - \frac{2A}{p^4} \tag{22}$$

---

*See Appendix A.

Substituting the initial conditions, the solution becomes

$$x(t) = \frac{2A}{p^4} \; [\cos pt - 1] + \frac{A}{p^2} t^2 \; , \tag{23}$$

which is the complete description of the motion of the valve head in the first half of the rise based on the one degree of freedom system.

Operational Calculus Analysis

To apply the method developed in Chapter III, it is necessary to have the operational transfer function

$$G(s) = \frac{P(s)}{Q(s)}$$

os the system. From $G(s)$, all the terms in equation (15), Chapter III, can be prepared.

Referring to Figure 7, the equation of motion becomes

$$MD^2x + Kx = K_L \; Z(t) \; . \tag{24}$$

Or, in operational notation

$$(Ms^2 + K) \; x = f(t) \; , \tag{25}$$

where $s = d/dt$ and $f(t) = K_L Z(t)$. Hence,

$$x(t) = \left[\frac{1}{Ms^2 + K}\right] f(t) \; , \tag{26}$$

and the operational transfer function is

$$G(s) = \frac{P(s)}{Q(s)} = \frac{1}{Ms^2 + K} \; . \tag{27}$$

The roots of $Q(s) = 0$ are

$$s_1 = \pm j\sqrt{\frac{K}{M}} \ , \tag{28}$$

a pair of complex conjugates of the form $s_k = a_k \pm jb_k$. Therefore

$$|s_1| = \sqrt{a_1^2 + b_1^2} = +\sqrt{\frac{K}{M}} \tag{a}$$

$$\theta_1 = \arctan(b_1/a_1) = \arctan\frac{\sqrt{K/M}}{0} = \frac{\pi}{2} \tag{b}$$

$$Q'(s) = \left.\frac{dQ(s)}{ds}\right|_{s=s_1} = 2Ms_1 = j\,2\sqrt{KM} \tag{c}$$

$$P(s_1) = 1 \tag{d}$$

$$\left|\frac{P(s_1)}{Q'(s_1)}\right| = \left|\frac{1}{j\,2\sqrt{KM}}\right| = \left|-j\,1/2\sqrt{KM}\right| = \frac{1}{2\sqrt{KM}} \tag{e}$$

$$\varphi_1 = \arctan\frac{-1/2\sqrt{KM}}{0} = -\frac{\pi}{2} \tag{f}$$

For a polynomial input the last term in equation (15), Chapter III, becomes zero, and equation (15) takes the form

$$x(t) = 2\sum_{k=p}^{r}\sum_{n=o}^{n}\left|\frac{P(s_k)}{Q'(s_k)}\right|\frac{e^{-a_k t}}{|s_k|^{n+1}}\cos\left\{b_k t + \varphi_k - (n+1)\theta_k\right\}\cdot f^{(n)}(0)$$

$$-\,2\sum_{k=p}^{r}\sum_{n=o}^{n}\left|\frac{P(s_k)}{Q'(s_k)}\right|\frac{1}{|s_k|^{n+1}}\cos\left\{\varphi_k - (n+1)\theta_k\right\} f^{(n)}(t) \tag{29}$$

Substituting equations (a) through (f) into equation (29)

$$x(t) = \frac{1}{\sqrt{KM}} \sum_{n=o}^{n} \frac{1}{\sqrt{\frac{K}{M}}^{\,n+1}} \cos\left\{\sqrt{\frac{K}{M}}\, t - (n+2)\,\frac{\pi}{2}\right\} f^{(n)}(0)$$

$$- \frac{1}{\sqrt{KM}} \sum_{n=o}^{n} \frac{1}{\sqrt{\frac{K}{M}}^{\,n+1}} \cos\left\{-(n+2)\,\frac{\pi}{2}\right\} \cdot f^{(n)}(t) \qquad (30)$$

which is the response of the system in Figure 7 to any polynomial input of the form

$$f(t) = \sum_{n=o}^{n} A_n t^n \,. \qquad (31)$$

For the first half of the rise of a parabolic cam curve, equation (31) becomes

$$f(t) = \frac{K_L H}{2\beta^2}\, t^2 \qquad (32)$$

Substituting equation (32) into equation (30) the response becomes

$$x(t) = \frac{2A}{p^4}\,[\cos pt - 1] + \frac{A}{p^2}\, t^2 \,. \qquad (33)$$

It can be readily seen that equations (33) and (23) are identical. This response can be differentiated repeatedly to provide velocity and acceleration information.

Computer Analysis

The one degree of freedom system can be promptly set into the Runge-Kutta based computer program described in Chapter II. Since

$K_L = 2.2104 \times 10^4$ lb/in, $K_s = 296$ lb/in, $K = 3.24 \times 10^4$ lb/in, and $M = 3.286 \times 10^{-3}$ lb-sec^2/in, the equation of motion can be written in the form

$$\frac{d^2x}{dt^2} = \frac{3.2106 \times 10^4}{3.286 \times 10^{-3}} Z - \frac{3.24 \times 10^4}{3.286 \times 10^{-3}} x .$$

Now, defining $X(1) = x$ and $X(2) = \frac{dx}{dt}$, having increments given by $\frac{d}{dt} X(1) = X(2)$ and $\frac{d}{dt} X(2) = \frac{d^2x}{dt^2}$, the increments of the output variable, in the language of the program, are expressed as:

```
D(1,K) = (DT)(X(2))                                          $
D(2,K) = (DT)(((3.2104**4)Z - (3.24**4)X(1))/(3.286**-3))    $
```

A sample printout of the program for the one degree of freedom system for a complete rise-dwell-return motion is shown beginning on page 69 of Appendix B. The initial conditions x and $dx/dt = 0$ are expressed by the array statement:

```
ARRAY X(2) = (0.0,0.0)                                       $
```

For a description of the forcing function $Z(t)$, which is expressed in the program by the logical group beginning at memory location 0238. See Appendix A. Notice the subsidiary information that can be printed out immediately. For example, velocity is $X(2)$, acceleration is d^2x/dt^2, and the deviation at the response from the cam lift is given by

```
DIFF = Z - X(1)                                              $
```

The results of this program, for a maximum rise of 0.6 inches in 120° of cam rotation at about 1860 camshaft revolutions per minute, are given in Figures 9, 10, and 11 in Chapter VII.

The camshaft speed is related to β, or time, by

$$N = \frac{1}{6} \frac{\Delta\theta}{\Delta t} ,$$

where $\Delta\theta$ is the number of camshaft degrees of rotation passed through in the time interval Δt seconds, and N is the cam speed in revolutions per minute.

There is no need to tabulate the output of this program versus the closed form solutions because the accuracy of the program has already been demonstrated in Chapter II. Such a tabulation will be made in the next chapter on the analysis of the many degree of freedom system.

CHAPTER V

ANALYSIS OF THE MANY DEGREE OF FREEDOM SYSTEM

Now that all of the necessary ground work has been laid and a basis of comparison established, the analysis of the many degree of freedom system representing a typical overhead valve mechanism shown in Figure 1 can be accomplished. The linkage is again represented by a lumped parameter system, the evolution of which is depicted in Figures 4, 5, and 6 in Chapter IV. The only difference is that the system to be analyzed is now the four degree of freedom system shown in Figure 6 instead of the one degree of freedom approximation in Figure 7.

Operational Calculus Analysis

In order to treat this four degree of freedom system by the methods of Chapter III it is only necessary to determine the operational transfer function of the system in the form

$$G(s) = \frac{P(s)}{Q(s)} .$$

By Newtonian methods, one can quite readily write the equations of motion for the system as follows:

$$\begin{aligned}
m_1 s^2 x_1 &= -x_1 k_s + (x_2 - x_1) k_{12} \\
m_2 s^2 x_2 &= -(x_2 - x_1) k_{12} + (x_3 - x_2) k_{23} \\
m_3 s^2 x_3 &= -(x_3 - x_2) k_{23} + (x_4 - x_3) k_{34} \\
m_4 s^2 x_4 &= -(x_4 - x_3) k_{34} + (Z - x_4) k_{45} ,
\end{aligned} \tag{34}$$

where $s = d/dt$ is the differential operator. At this point define

$$\Gamma_n = \left\{ m_n s^2 + k_{(n-1),n} + k_{n,(n+1)} \right\} , \tag{35}$$

where $k_{0,1} = k_s$ is a special property. Using the notation of equation (35) the equations of motion become

$$\begin{aligned}
\Gamma_1 x_1 - k_{12} x_2 &= 0 \\
-k_{12} x_1 + \Gamma_2 x_2 - k_{23} x_3 &= 0 \\
-k_{23} x_2 + \Gamma_3 x_3 - k_{34} x_4 &= 0 \\
-k_{34} x_3 + \Gamma_4 x_4 &= f(t) ,
\end{aligned} \tag{36}$$

where $f(t) = k_{45} Z(t)$ is the input signal. Since $G(s)$ is defined to be the operational relationship between the input and output signals,

$$x(t) = G(s)\, f(t) ,$$

equations (36) can be solved by Cramer's rule for x_1 as being

$$x_1(t) = \frac{\begin{vmatrix} 0 & -k_{12} & 0 & 0 \\ 0 & \Gamma_2 & -k_{23} & 0 \\ 0 & -k_{23} & \Gamma_3 & -k_{34} \\ f(t) & 0 & -k_{34} & \Gamma_4 \end{vmatrix}}{\begin{vmatrix} \Gamma & -k_{12} & 0 & 0 \\ -k_{12} & \Gamma_2 & -k_{23} & 0 \\ 0 & -k_{23} & \Gamma_3 & -k_{34} \\ 0 & 0 & -k_{34} & \Gamma_4 \end{vmatrix}} . \tag{37}$$

Due to the fortunate arrangement of the zeroes in the determinants in equation (37), they can be expanded very quickly by cofactors to give

$$x_1(t) = \frac{k_{12}k_{23}k_{34} \cdot f(t)}{\Gamma_1\Gamma_2\Gamma_3\Gamma_4 - \Gamma_1\Gamma_2 k_{34}^2 - \Gamma_1\Gamma_4 k_{23}^2 - \Gamma_3\Gamma_4 k_{12}^2 + k_{12}^2 k_{34}^2} \quad (38)$$

By using the notation

$$\begin{aligned} k_1 &= k_s + k_{12} \\ k_2 &= k_{12} + k_{23} \\ k_3 &= k_{23} + k_{34} \\ k_4 &= k_{34} + k_{45} \end{aligned} \quad (a)$$

and replacing the values Γ_n with equation (35) the denominator of equation (38) becomes

$$Q(s) = \left[m_1 m_2 m_3 m_4\right] s^8 + \begin{bmatrix} m_1 m_2 m_3 k_4 \\ +m_1 m_2 k_3 m_4 \\ +m_1 k_2 m_3 m_4 \\ +k_1 m_2 m_3 m_4 \end{bmatrix} s^6 + \begin{bmatrix} m_1 m_2 k_3 k_4 \\ +m_1 k_2 m_3 k_4 \\ +m_1 k_2 k_3 m_4 \\ +k_1 m_2 k_3 m_4 \\ +k_1 m_2 m_3 k_4 \\ -m_1 m_2 k_{34} \\ -m_1 m_4 k_{23}^2 \\ -m_3 m_4 k_{12}^2 \end{bmatrix} s^4$$

$$\begin{bmatrix} m_1k_2k_3k_4 \\ +k_1m_2k_3k_4 \\ +k_1k_2m_3k_4 \\ +k_1k_2k_3m_4 \\ -(m_1k_2+k_1m_2)k_{34}^2 \\ -(m_1k_4+k_1m_4)k_{23}^2 \\ -(m_3k_4+k_3m_4)k_{12}^2 \end{bmatrix} s^2 + \begin{bmatrix} k_1k_2k_3k_4 \\ -k_1k_2k_{34}^2 \\ -k_1k_4k_{23}^2 \\ -k_3k_4k_{12}^2 \\ +k_{12}^2k_{34}^2 \end{bmatrix}, \tag{39}$$

and the numerator becomes

$$P(s) = k_{12}k_{23}k_{34} \ . \tag{40}$$

By arranging the coefficients of equation (39) in the arrays as shown, recursive patterns appear, which facilitate checking.

At this point it is expedient to substitute the numerical values for the masses and stiffness values as shown in Figure 6. A good desk calculator from here on would be very useful. This done, equations (39) and (40) become

$$Q(s) = (0.272998 \times 10^{-12})\ s^8 + (0.364998 \times 10^{-3})\ s^6 + (0.15166 \times 10^6)\ s^4 + (0.186685 \times 10^{14})\ s^2 + (0.176728 \times 10^{21}) \tag{41}$$

and

$$P(s) = 1.480 \times 10^{15} \ . \tag{42}$$

The next step is to prepare each of the terms necessary for substitution into the general equation for the response,

$$x_1(t) = 2 \sum_{k=p}^{r} \sum_{n=o}^{n} \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{e^{a_k t}}{|s_k|^{n+1}} \cos \left\{ b_x t + \varphi_k - (n+1)\theta_k \right\} f^{(n)}(0)$$

$$2 \sum_{k=p}^{r} \sum_{n=o}^{n} \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{1}{|s_k|^{n+1}} \left\{ \cos \varphi_k - (n+1)\theta_k \right\} f^{(n)}(t), \quad (43)$$

where the last term is not shown since it is zero for a polynomial input. There will be a set of terms for each of the four pairs of complex conjugate roots associated with equation (41). At first glance, finding these roots appears a formidable task. However, a very accurate first approximation to one pair of the roots is obtained by taking only the last two terms in equation (41). That is

$$(0.186685 \times 10^{14})\ s^2 + (0.176728 \times 10^{21}) \approx 0$$

from which $s_{approx.} = \pm\ j\ 3090$. By Newton's method this first approximation can be quickly refined, and by synthetic division the order of the equation $Q(s) = 0$ can be reduced. Thus the other roots can be found. The roots of $Q(s) = 0$, therefore are

$$\begin{aligned} s_1 &= \pm\ j\ 3190.43 \\ s_2 &= \pm\ j\ 14214.3 \\ s_3 &= \pm\ j\ 22882.1 \\ s_4 &= \pm\ j\ 24519.1 \end{aligned} \qquad (b)$$

Associated with these four pairs of roots are four modes of vibration having periods of

$$\lambda_1 = 0.00197 \text{ sec.}$$

$$\lambda_2 = 0.000442 \text{ sec.}$$

$$\lambda_3 = 0.000274 \text{ sec.} \qquad \text{(c)}$$

$$\lambda_4 = 0.000255 \text{ sec.} ,$$

which are easily verified by the computer analysis which follows later. See Figures 9, 10, and 11 in Chapter VII. Also for the example at hand

$$Q'(s_k) = \frac{dQ(s_k)}{ds} = (2.18398 \times 10^{-12})\ s_k^{\ 7} + (2.18999 \times 10^{-3})\ s_k^{\ 5}$$

$$+ (0.60664 \times 10^{6})\ s_k^{\ 3} + (0.37737 \times 10^{14})\ s_k \qquad (44)$$

For the four pairs of roots (b) the terms for substitution into equation (43) are given in Table 2.

Table 2. Terms for Substitution into Equation (43)

k	s_k	θ_k	$Q'(s_k)$	$\frac{P(s_k)}{Q'(s_k)}$	φ_k
1	3190	$\pi/2$	$+j\ 10.1292 \times 10^{16}$	0.01459	$-\pi/2$
2	14214.3	$\pi/2$	$-j\ 19.088 \times 10^{16}$	0.00775	$+\pi/2$
3	22882	$\pi/2$	$+j\ 16.017 \times 10^{16}$	0.00924	$-\pi/2$
4	24519	$\pi/2$	$-j\ 24.48 \times 10^{16}$	0.00604	$+\pi/2$

Substituting each of these quantities in Table 2 properly into equation (43) gives the complete response of the four degree of freedom system to any polynomial input excitation of the form,

$$f(t) = \sum_{n=o}^{n} A_n \, t^n \tag{45}$$

as being

$$
\begin{aligned}
x_1(t) = {} & 2 \sum_{n=o}^{n} \frac{(0.01459)}{(3190)^{n+1}} \cos\left\{3190\ t - \frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(0) \\
& + 2 \sum_{n=o}^{n} \frac{(0.00775)}{(14214.3)^{n+1}} \cos\left\{14214.3\ t + \frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(0) \\
& + 2 \sum_{n=o}^{n} \frac{(0.00924)}{(22882)^{n+1}} \cos\left\{22882\ t - \frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(0) \\
& + 2 \sum_{n=o}^{n} \frac{(0.00604)}{(24519)^{n+1}} \cos\left\{24519\ t + \frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(0) \\
& - 2 \sum_{n=o}^{n} \frac{(0.01459)}{(3190)^{n+1}} \cos\left\{-\frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(t) \\
& - 2 \sum_{n=o}^{n} \frac{(0.00775)}{(14214.3)^{n+1}} \cos\left\{+\frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(0) \\
& - 2 \sum_{n=o}^{n} \frac{(0.00924)}{(22882)^{n+1}} \cos\left\{-\frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(t) \\
& - 2 \sum_{n=o}^{n} \frac{(0.00604)}{(24519)^{n+1}} \cos\left\{+\frac{\pi}{2} - (n+1)\frac{\pi}{2}\right\} f^{(n)}(t) \,.
\end{aligned} \tag{46}
$$

For a parabolic cam curve (second order polynomial), in the first half of the rise, equation (45) becomes

$$f(t) = \frac{k_{45}H}{2\beta^2} t^2 ,$$

and equation (46) becomes

$$x_1(t) = \frac{2k_{45}H}{\beta^2} \cdot \left[\begin{aligned} & \frac{0.01459}{(3190)^3} \left\{\cos (3190\ t) \ -1\right\} \\ & - \frac{0.00775}{(14214)^3} \left\{\cos (14214\ t) -1\right\} \\ & + \frac{0.00924}{(22882)^3} \left\{\cos (22882\ t) -1\right\} \\ & - \frac{0.00604}{(24519)^3} \left\{\cos (24519\ t) -1\right\} \end{aligned} \right]$$

$$+ (0.8372 \times 10^{-5})(k_{45}H/2\beta^2)\ t^2 . \tag{48}$$

Equation (48) can be readily differentiated once and then again to give velocity and acceleration of the valve.

$$VEL = \frac{2k_{45}H}{\beta^2} \cdot \left[\begin{aligned} & - \frac{0.01459}{(3190)^2} \ \sin (3190\ t) \\ & + \frac{0.00775}{(14214)^2} \ \sin (14214\ t) \\ & - \frac{0.00924}{(22882)^2} \ \sin (22882\ t) \\ & + \frac{0.00604}{(24519)^2} \ \sin (24519\ t) \end{aligned} \right]$$

$$+ (1.6744 \times 10^{-5})(k_{45}H/2\beta^2)\ t \tag{49}$$

and

$$ACN = \frac{2k_{45}H}{\beta^2} \left[-\frac{0.01459}{3190} \cos(3190\ t) + \frac{0.00775}{14214} \cos(14214\ t) - \frac{0.00924}{22882} \cos(22882\ t) + \frac{0.00604}{24519} \cos(24519\ t) \right] + (1.6744 \times 10^{-5})(k_{45}H/2\beta^2) \tag{50}$$

It should be observed that this is an exact solution, for no approximations have been made beyond the lumped parameter system C in Figure 6. A comparison of equations (48), (49), and (50) with the computer solution can be seen in Table 3 on page 46.

Computer Analysis

The system in Figure 6 can be formulated for analysis by the Runge-Kutta based computer program presented in Chapter II in just two steps:

Step one: Write the equations of motion in the form

$$\begin{aligned}
\ddot{x} &= -\frac{k_s}{m_1} x_1 + (x_2 - x_1)\frac{k_{12}}{m_1} \\
\ddot{x}_2 &= -(x_2 - x_1)\frac{k_{12}}{m_2} + (x_3 - x_2)\frac{k_{23}}{m_2} \\
\ddot{x}_3 &= -(x_3 - x_2)\frac{k_{23}}{m_3} + (x_4 - x_3)\frac{k_{34}}{m_3} \\
\ddot{x}_4 &= -(x_4 - x_3)\frac{k_{34}}{m_4} + (Z - x_4)\frac{k_{45}}{m_4} \quad ,
\end{aligned} \tag{51}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$.

Step two:

Define $\begin{bmatrix} X(1) = x_1 \\ X(2) = \dot{x}_1 \\ X(3) = x_2 \\ X(4) = \dot{x}_2 \\ X(5) = x_3 \\ X(6) = \dot{x}_3 \\ X(7) = x_4 \\ X(8) = \dot{x}_4 \end{bmatrix}$, so that $\begin{bmatrix} \dot{X}(1) = X(2) \\ \dot{X}(2) = \ddot{x}_1 \\ \dot{X}(3) = X(4) \\ \dot{X}(4) = \ddot{x}_2 \\ \dot{X}(5) = X(6) \\ \dot{X}(6) = \ddot{x}_3 \\ \dot{X}(7) = X(8) \\ \dot{X}(8) = \ddot{x}_4 \end{bmatrix}$

Now the increments in the output variable x_i can be written in the language of the computer program as being

```
D(1,K) = (DT)(X(+))                                                $
D(2,K) = (DT)(((-296.0)X(1) + (X(3)-X(1))(30**4))/1.333**-3)       $
D(3,K) = (DT)(X(4))                                                $
D(4,K) = (DT)((-(X(=)-X(1))(30**4) + (X(5)-X(3))(7.15**4))/
         (1.075**-3))                                              $
D(5,K) = (DT)(X(6))                                                $
D(6,K) = (DT)((-(X(5)-X(3))(7.15**4) + (X(7)-X(5))(6.90**4))/
         (0.486**-3))                                              $
D(7,K) = (DT)(X(8))                                                $
D(8,K) = (DT)((-(X(7)-X(5))(6.90**4) + (Z-X(7))(11.8**4))/
         (0.392**-3))                                              $
```

The forcing function $Z(t)$ is again given by the logical set on page 64 of Appendix A.

As in the previous chapter, the print-out can include any of the variables -- x_i, dx_i/dt, d^2x_i/dt^2, $i=1,2,3,4$ -- that are desired. Since, in this case, the motion of the valve is desired, the output is given by t, x_1, $\dot{x}_1$, and $\ddot{x}_1$. The complete program begins on page 73 of Appendix B. The results for a complete cycle of motion (rise-dwell-return) in 120° of camshaft rotation corresponding to 1860 camshaft RPM, are given following the program in Appendix B, and are plotted in Figures 9, 10, and 11 in Chapter VII.

An immediate byproduct here is the reaction force on the cam surface, which is in essence given by

$$\text{CAMFORCE} = k_{45}\ (Z-x_4)\ . \tag{52}$$

As will be pointed out later, certain modifications to equation (52) willbe necessary. This reaction force at the cam is the topic of the next chapter.

Of course, values of Z and x_4 have been printed out by the computer program for purposes of checking and to obtain data for the plotting of Figures 9, 10, and 11 in Chapter VII.

Comparison of Computer and Analytical Solutions

As a check on the accuracy of the two methods of analyzing the many degree of freedom system, it would be well to compare values of displacement, velocity, and acceleration of the valve at certain times, calculated by each of the methods. This comparison is shown in Table 3 for the first half of the rise of a parabolic cam curve. The maximum rise is $H = 0.6$ inches, and the complete motion is a rise-dwell-return in 120° of camshaft rotation at 1860 RPM.

Table 3. Comparison of the Computer and Analytical Solutions for Lift, Velocity, and Acceleration

t (sec.)	Lift		Velocity		Acceleration	
	analytical	computer	analytical	computer	analytical	computer
T	X	X	VEL	VEL	ACN	ACN
.000100	.000000	.000000	.000675	.000202	10.0900	17.7377
.000200	.000001	.000001	.066923	.071832	2835.22	2879.11
.000300	.000053	.000054	1.48010	1.48905	33563.7	33604.9
.000400	.000482	.000484	8.34505	8.35908	106245	106304
.000500	.001941	.001944	21.4645	21.4844	144831	144888
.000600	.004819	.004825	36.2587	36.2853	155938	156024
.000700	.009304	.009314	54.3932	54.4302	210833	210944
.000800	.015874	.015887	77.5625	77.6088	245661	245733
.000900	.024897	.024915	103.373	103.426	274121	274192
.001000	.036647	.036671	131.768	131.828	280833	280899
.001100	.051142	.051172	157.189	157.254	227969	228003
.001200	.067985	.068022	179.888	179.956	238309	238343
.001300	.087190	.087234	204.030	204.099	226835	226811
.001400	.108600	.108650	222.889	222.953	153289	153231
.001500	.131594	.131650	236.506	236.506	121573	121556
.001600	.155762	.155824	245.745	245.802	57101.9	57054.7
.001700	.180545	.180612	249.576	249.627	36788.8	36728.1
.001800	.205725	.205798	254.181	254.228	42259.3	42248.8
.001900	.231242	.231319	254.854	254.899	-30713.6	-30750.5
.002000	.256537	.256618	251.213	251.259	-22594.9	-22525.1
.002100	.281624	.281710	251.086	251.142	10108.2	10198.6

CHAPTER VI

REACTION FORCE AT THE CAM

As was pointed out in Chapter I, very little has been done analytically on the reaction force at the cam. Due to the nature of both of the methods of analysis presented in this study, the reaction force is immediately available by taking the output of the system to be x_4, as is shown in Figure 8, instead of x_1, as is shown in Figure 3. Then the reaction force at the cam is given by

$$\text{CAMFORCE} = k_{45}(Z - x_4) \quad . \tag{52}$$

There is an immediate drawback to the method suggested by equation (52). That is, many significant figures must be taken to insure accuracy, since Z is so nearly equal to x_4. However, with only four or five significant figures good "ball park" results should be expected. There are other necessary modifications depending on the nature of the system at hand. For the overhead valve linkage in Figure 6, these modifications concern the case of a terminal mass, m_5, at the cam surface

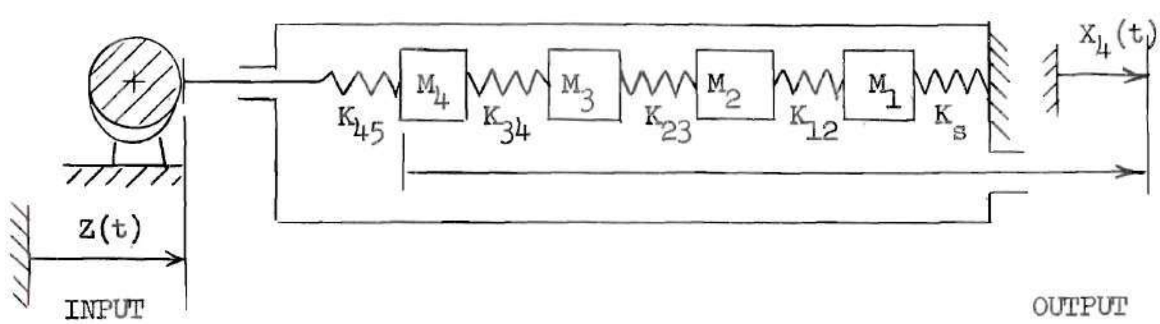


Figure 8. Equivalent System for Cam Force

and transposition of the result into the cam datum. These will be discussed further later.

Operational Calculus Analysis

The only unknown term in equation (52) is x_4 which can be readily determined by the methods of Chapters III and V. From the equations of motion (36) and by Cramer's rule again

$$x_4(t) = \frac{\begin{vmatrix} \Gamma_1 & -k_{12} & 0 & 0 \\ -k_{12} & \Gamma_2 & -k_{23} & 0 \\ 0 & -k_{23} & \Gamma_3 & 0 \\ 0 & 0 & -k_{34} & f(t) \end{vmatrix}}{\begin{vmatrix} \Gamma_1 & -k_{12} & 0 & 0 \\ -k_{12} & \Gamma_2 & -k_{23} & 0 \\ 0 & -k_{23} & \Gamma_3 & -k_{34} \\ 0 & 0 & -k_{34} & \Gamma_4 \end{vmatrix}} . \tag{53}$$

Expanding the determinants of equation (53) by cofactors,

$$x_4(t) = \frac{\Gamma_1\Gamma_2\Gamma_3 - \Gamma_1 k_{23}^2 - \Gamma_3 k_{12}^2 \cdot f(t)}{\Gamma_1\Gamma_2\Gamma_3\Gamma_4 - \Gamma_1\Gamma_2 k_{34}^2 - \Gamma_1\Gamma_4 k_{23}^2 - \Gamma_3\Gamma_4 k_{12}^2 - k_{12}^2 k_{34}^2} , \tag{54}$$

where the denominator is the same as in equation (38).

Therefore

$$P(s) = \Gamma_1\Gamma_2\Gamma_3 - \Gamma_1 k_{23}^2 - \Gamma_3 k_{12}^2 \tag{55}$$

$$= \begin{bmatrix} m_1 m_2 m_3 \end{bmatrix} s^6 + \begin{bmatrix} m_1 m_2 k_3 \\ +m_1 k_2 m_3 \\ +k_1 m_2 m_3 \end{bmatrix} s^4 + \begin{bmatrix} m_1 k_2 k_3 \\ +k_1 m_2 k_3 \\ +k_1 k_2 m_3 \\ -m_1 k_{23} \\ -m_3 k_{12}^2 \end{bmatrix} s^2 + \begin{bmatrix} k_1 k_2 k_3 \\ -k_1 k_{23} \\ -k_3 k_{12}^2 \end{bmatrix},$$

and $Q(s)$ is given by equation (39) in Chapter V. Therefore x_4 is given by the general equation for the response of a linear system forced by a polynomial input,

$$x_4(t) = 2 \sum_{k=p}^{r} \sum_{n=o}^{n} \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{e^{a_k t}}{|s_k|^{n+1}} \cos\left\{ b_k t + \varphi_k - (n+1)\theta_k \right\} f^{(n)}(0)$$

$$- 2 \sum_{k=p}^{r} \sum_{n=o}^{n} \left| \frac{P(s_k)}{Q'(s_k)} \right| \frac{1}{|s_k|^{n+1}} \cos\left\{ \varphi_k - (n+1)\theta_k \right\} f^{(n)}(t), \tag{56}$$

for which each of the terms necessary for substitution are given in Table 4.

Table 4. Terms for Substitution into Equation (56)

k	s_k	θ_k	$Q'(s_k)$	$\frac{P(s_k)}{Q'(s_k)}$	φ_k
1	14214.3	$\pi/2$	$-j\ 190.88 \times 10^{16}$	0.01962	$-\pi/2$
2	3190	$\pi/2$	$+j\ 10.1292 \times 10^{16}$	0.00345	$-\pi/2$
3	22882	$\pi/2$	$+j\ 16.017 \times 10^{16}$	0.02314	$-\pi/2$
4	24519	$\pi/2$	$-j\ 24.48 \times 10^{16}$	0.01851	$-\pi/2$

For a parabolic cam curve given by

$$f(t) = \frac{k_{45}H}{2\beta^2}\, t^2 ,$$

the response x_4 becomes

$$x_4(t) = \frac{2k_{45}H}{\beta^2}\left[\frac{0.00345}{(3190)^3}\left\{\cos\ (3190\ t)\ -1\right\} + \frac{0.01962}{(14214)^3}\left\{\cos\ (14214\ t)\ -1\right\} + \frac{0.02314}{(22882)^3}\left\{\cos\ (22882\ t)\ -1\right\} + \frac{0.01851}{(24519)^3}\left\{\cos\ (24519\ t)\ -1\right\}\right]$$

$$+\ (0.8436 \times 10^{-5})(k_{45}H/2\beta^2)\ t^2 . \tag{57}$$

Now, upon substituting equation (57) into equation (52) the reaction force becomes

$$\text{CAMFORCE} = (11.7999 \times 10^4)(k_{45}H/2\beta^2)\ t^2 \tag{58}$$

$$+\ (47.2)(k_{45}H/2\beta^2)\left[\frac{0.00345}{(3190)^3}\left\{1 - \cos\ (3190\ t)\right\} + \frac{0.01962}{(14214)^3}\left\{1 - \cos\ (14214\ t)\right\} + \frac{0.02314}{(22882)^3}\left\{1 - \cos\ (22882\ t)\right\} + \frac{0.01851}{(24519)^3}\left\{1 - \cos\ (24519\ t)\right\}\right]$$

Equation (58) is a terminating cosine series demonstrating four modes of vibration of the linkage and is a quite good representation of what really happens. For example, if the overhead valve linkage were analyzed as a continuum, rather than a lumped parameter system, (see Appendix D) the expression for the reaction force would be an infinite cosine series of the form

$$\text{CAMFORCE} = Ct^2 + \sum_{i=1}^{\infty} K_i \left\{ 1 - \cos\,(b_i t) \right\}, \tag{59}$$

where C is a constant, and K_i and b_i are constant functions of the eigenvalues p_i resulting from the roots of

$$\cot p_i = \frac{p_i}{\eta} - \frac{1}{\mu p_i}. \tag{a}$$

Equations (a) and (59) are derived in Appendix D. Notice that equation (58) can be taken to be the first four terms of equation (59), with the coefficients possibly modified to encourage more rapid convergence.

Computer Analysis

Having x_4 available in the computer memory, it is only necessary to instruct the computer to print it out. Further, the cam force can be readily computed by inserting one card into the already existing program at memory location 0599 in the program beginning on page 73 of Appendix B. The results of the computer analysis and the analytical solution (equation (58)) are compared for the first half of the rise in Table 5.

Table 5. Camforce by Operational Calculus Method and Runge-Kutta Computer Program

t(sec)	CAMFORCE (lbs.)	
	Analytical	Computer
T	CMF	CAMFORCE
.000100	83.1526	83.1288
.000200	159.103	158.921
.000300	226.279	225.828
.000400	324.944	323.846
.000500	432.774	431.184
.000600	593.911	592.164
.000700	716.596	714.153
.000800	755.584	752.841
.000900	809.595	806.639
.001000	847.612	844.336
.001100	842.712	839.982
.001200	779.013	776.171
.001300	654.043	651.322
.001400	560.025	558.204
.001500	481.054	479.346
.001600	359.551	358.617
.001700	233.863	233.399
.001800	155.271	154.723
.001900	142.178	142.573
.002000	161.104	161.396
.002100	177.969	178.037

Modifications to Camforce Analysis

Terminal Mass at the Cam

In Figure 6, Chapter IV, the system is shown to have a terminal mass at the cam surface, which does not affect the vibrational characteristics of the system because it follows the cam surface exactly during normal operation. However, its inertia force affects the reaction force at the cam surface, thus causing equation (52) to be modified to

$$\text{CAMFORCE} = k_{45}\ (Z - x_4) + m_5 \frac{d^2Z}{dt^2}\ . \tag{60}$$

Equation (60) is the expression used in the computer analysis and in the evaluation of the analytical solution. Table 5 compares the two solutions.

When the cam force becomes negative it can be considered that the follower has left the surface of the cam. This phenomenon is called jump. In such a case these calculations are invalid because the system becomes a free vibration problem and is governed by a different set of equations of motion. This situation is not serious, though, since it is avoided in the design process. However, if the follower were fastened to the cam surface, then the negative reaction force would be meaningful. In this case there would be some constant force that could be superposed on the system so that the cam force never becomes negative. This constant force can be taken to be the amount of precompression necessary in the valve spring to prevent jump.

Transposition of the Cam Force into the Valve Datum

Since there are two levers in the actual overhead valve linkage, the reaction force at the cam itself would be the cam force in the equivalent system C multiplied by the lever ratio R used in the evolution of system C from system A. See Figures 4 through 6. However, for determining the necessary precompression in the valve spring, the reaction force in the equivalent system C is necessary because the valve spring is in the valve datum, about which system C is built.

CHAPTER VII

DISCUSSION OF RESULTS AND RECOMMENDATIONS

Discussion of Results

An entire cycle of rise-dwell-return motion has been analyzed by the computer method for the one degree of freedom approximation and the four degree of freedom system, both for the same cam profile and speed. The lift, velocity, and acceleration responses of the two systems are shown in Figures 9, 10, and 11. Also the equivalent cam lift, Z; velocity, dZ/dt; and acceleration, d^2Z/dt^2, are superposed on these figures. Therefore it is possible to see the deviation of the two systems from one another, as well as from the theoretical input signals. Notice that the one degree of freedom approximation is very good with respect to displacement, but leaves something to be desired concerning velocity and acceleration responses. To know the velocity of the valve head accurately is important, especially on the closing ramp. Poor knowledge of the velocity on the closing ramp can lead to false conclusions concerning the valve bounding off its seat upon closing. It is unnecessary to expound on the need to know accurately the acceleration values because stresses in the system are proportional to the accelerations of the links.

The reaction force calculated by the computer program for the four degree of freedom system indicates very large precompression forces necessary in the valve spring. This may seem further to point out the poorness of a parabolic cam curve in actual practice. The value indicated

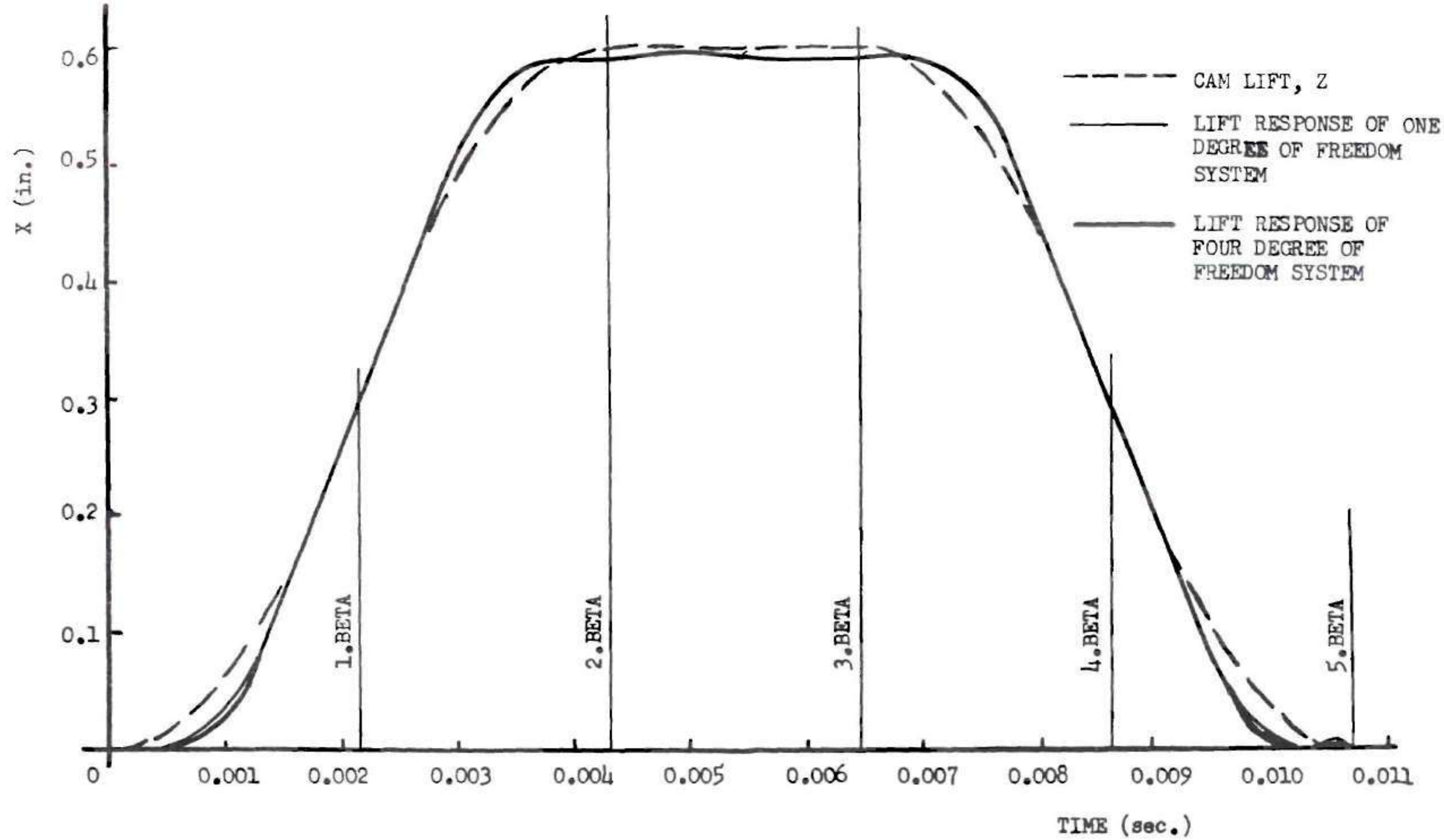


Figure 9. Lift Response of One and Four Degree of Freedom Systems.

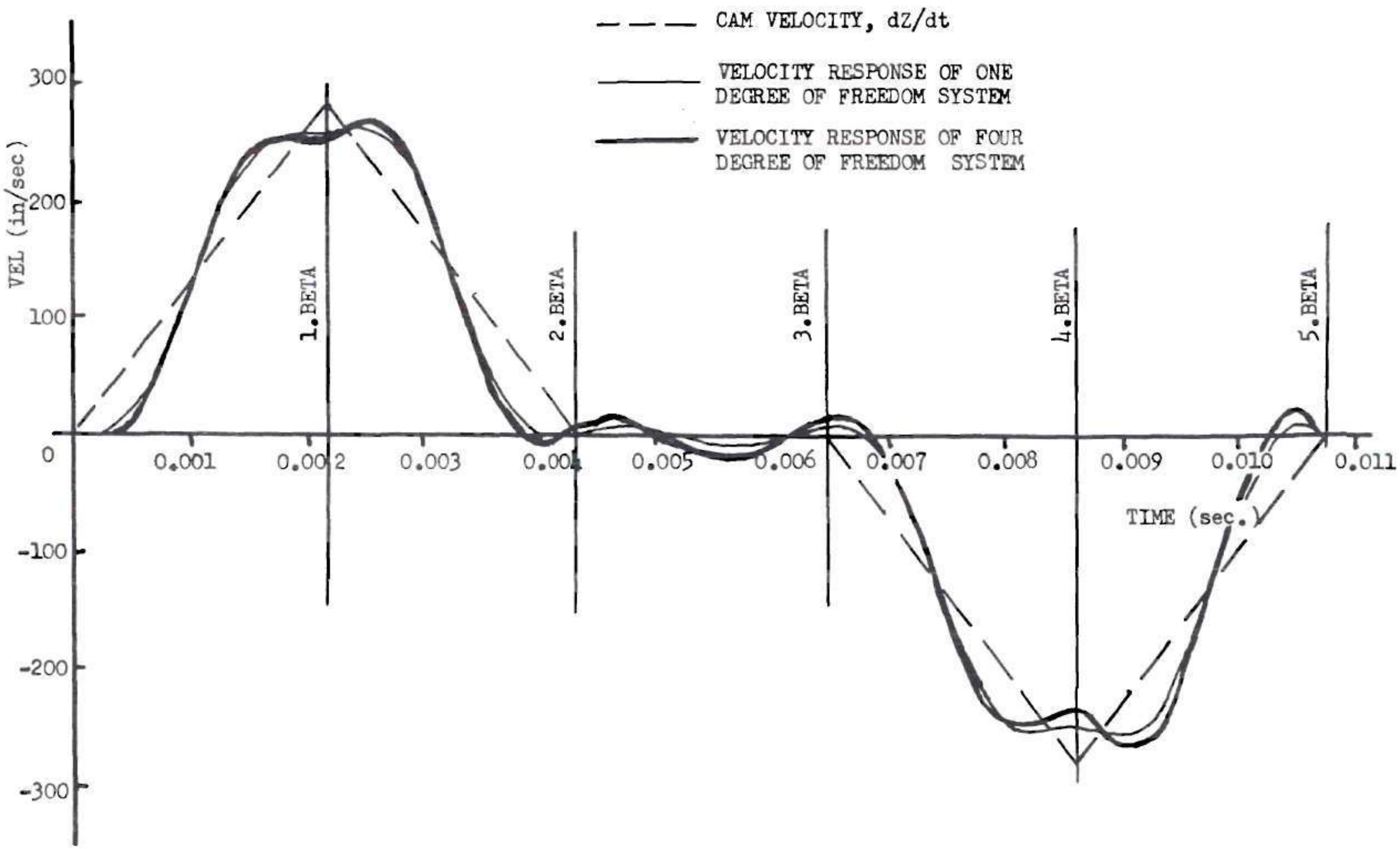


Figure 10. Velocity Response of One and Four Degree of Freedom Systems.

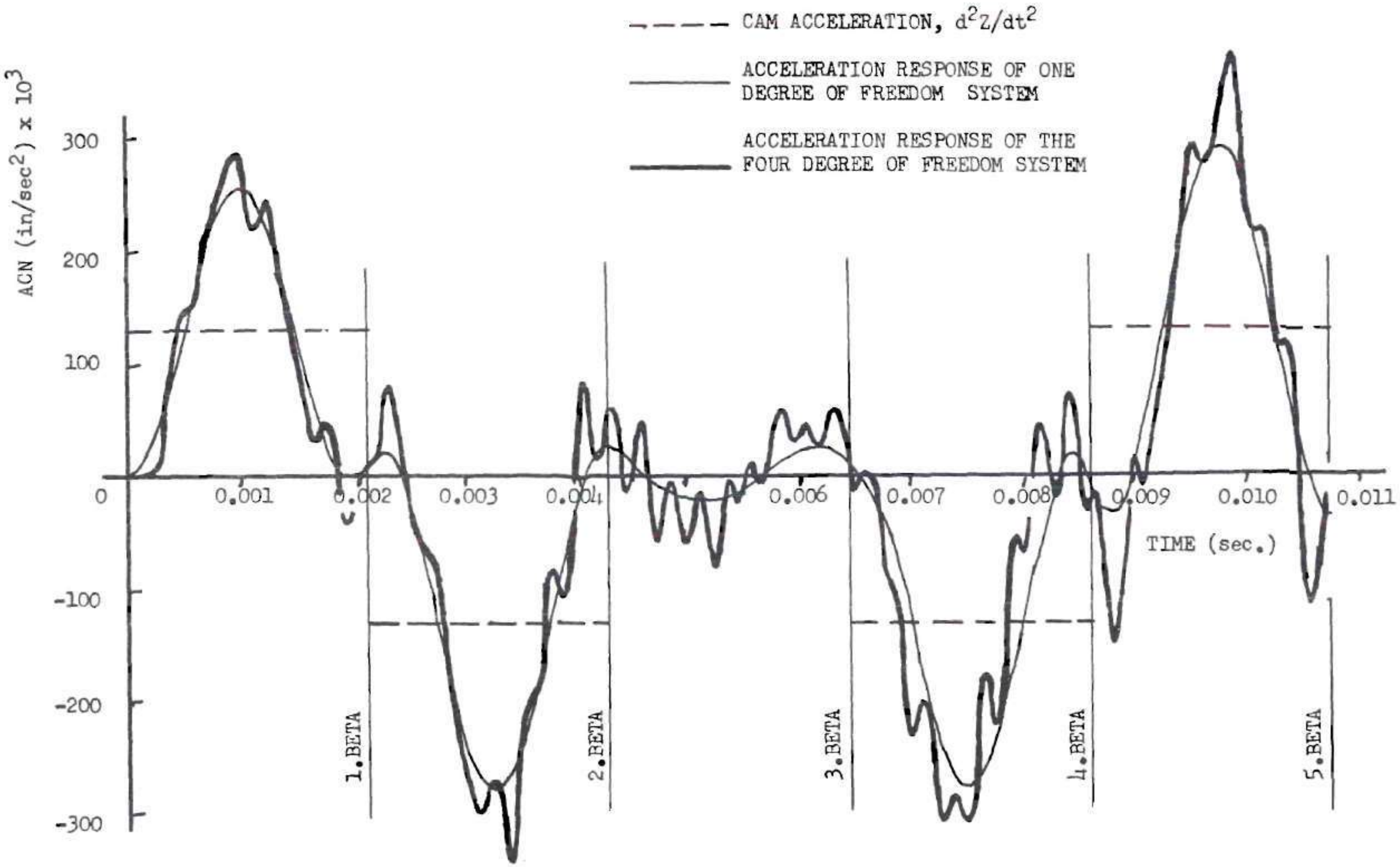


Figure 11. Acceleration Response of the One and Four Degree of Freedom Systems.

(beginning on page 73) is about eight times that used by Barkan[6] in his experimental work. Of course, the cam profile used in this example, besides the fact that there are step changes in the input acceleration, is quite radical too. That is to say, the dwell is very long, and little time is allowed for opening and closing the valve.

Some Other Useful Features of the Runge Kutta Program

It is possible in effect, to vary the cam speed in the analysis by converting the whole program into a loop and running it over and over again for several values of β. This is done by the logical set:

```
        L  =        $
        IV =        $        FV =      $
        FOR BETA = (IV, (FV - IV)/(L-1), FV)      $
BEGIN   WRITE ($$ANS1,FMT1)                        $
            -----
        (the rest of the program)
            -----
OUTPUT  ANS1 (BETA)                                $
FORMAT  FMT1 (*FOR BETA = *, S8.5,W3,WO)           $
            -----
        (the other FORMAT statements in the program
                  and other instructions)
            -----
        END                                        $
```

L is the number of different values of β in between its initial value, IV, and final value, FV. Also in doing this, output instructions are provided to print out the value of β at the beginning of each run. This procedure is a very powerful tool in analyzing the behavior of the system in the neighborhood of a resonant speed.

Recommendations

It is recommended that some experimental work be done in three areas:

1. verification of the lift, velocity, and acceleration responses in some real system predicted by the methods presented in this thesis.

2. verification of the predicted preload values necessary to prevent jump in some real system.

3. verification of the actual reaction forces predicted at the cam surface in some real system.

With the advent of modern high speed photography techniques, and dynamically reliable strain gages and recorders, none of the three areas of research mentioned above should be beyond the reach of a well equipped industrial or university laboratory.

APPENDIX A

PARABOLIC RISE - DWELL - RETURN MOTION

The cam curve used in all the examples in this work was the parabolic, or gravity, curve. Even though this curve is not the best for high speed cams because of the finite jumps in its acceleration characteristics, it was used in this work because it is the simplest polynomial curve available that retains all the mathematical characteristics normally found in higher order polynomials necessary for the demonstrations in this thesis. The general form of the curve is given by

$$Z(t) = a_o + a_1 t + a_2 t^2 \tag{61}$$

and the problem here is to determine a_o, a_1, and a_2 for the particular region of rise, dwell, or return. This is done by considering the end conditions (displacement and velocity) at the end of one section to be the same as those at the beginning of the next section as follows: See Figure 12.

<u>$0 \leq t \leq \beta$</u>

$$Z = \dot{Z} = 0 \quad \text{at} \quad t = 0$$

$$Z = \frac{H}{2} \quad \text{at} \quad t = \beta$$

Substituting these conditions into equation (61) results in three algebraic simultaneous equations, which can be solved readily for a_o, a_1, and a_2. Therefore

$$a_o = 0, \quad a_1 = 0, \quad \text{and} \quad a_2 = \frac{H}{2\beta^2},$$

so that

$$Z(t) = 2H\left[\frac{t}{2\beta}\right]^2. \tag{62}$$

<u>$\beta \leq t \leq 2\beta$</u>

$$\left.\begin{array}{l} Z = \frac{H}{2} \\ \dot{Z} = \frac{H}{\beta} \end{array}\right\} \quad \text{at} \quad t = \beta$$

$$\left.\begin{array}{l} Z = H \\ \dot{Z} = 0 \end{array}\right\} \quad \text{at} \quad t = 2\beta$$

Again proceeding as in the first region

$$a_o = -H, \quad a_1 = \frac{2H}{\beta}, \quad \text{and} \quad a_2 = -\frac{H}{2\beta^2},$$

so that

$$Z(t) = H\left[1 - 2\left(1 - \frac{t}{2\beta}\right)^2\right]. \tag{63}$$

<u>$2\beta \leq t \leq 3\beta$</u>

$$Z(t) = H \tag{64}$$

Obviously this region is the dwell.

<u>$3\beta \leq t \leq 4\beta$</u>

The $Z(t)$ equation will have the same form in this region as equation (63) with the input variable transformed to

$$\tau = 5\beta - t \ ,$$

so that

$$Z(\tau) = H \left[1 - 2\left(1 - \frac{\tau}{2\beta}\right)^2 \right] ,$$

or

$$Z(t) = H \left[1 - 2\left(1 - \frac{(5\beta - t)}{2\beta}\right)^2 \right] , \tag{65}$$

<u>$4\beta \leq t \leq 5\beta$</u>

The $Z(t)$ equation in this region will have the same form as equation (62) with the input variable again transformed to

$$\tau = 5\beta - t \ ,$$

so that

$$Z(t) = 2H \left[\frac{(5\beta - t)}{2\beta} \right]^2 . \tag{66}$$

Even though equations 63, 65, and 66 are not immediately seen to be in the form of equation (61), they can be reduced to that form. In the form shown these equations are easiest to check for the end conditions and are quite amenable to the computer language.

The Algol[10] formulation of expressions (2) through (6) is given below and also can be seen in the programs in Appendix B.

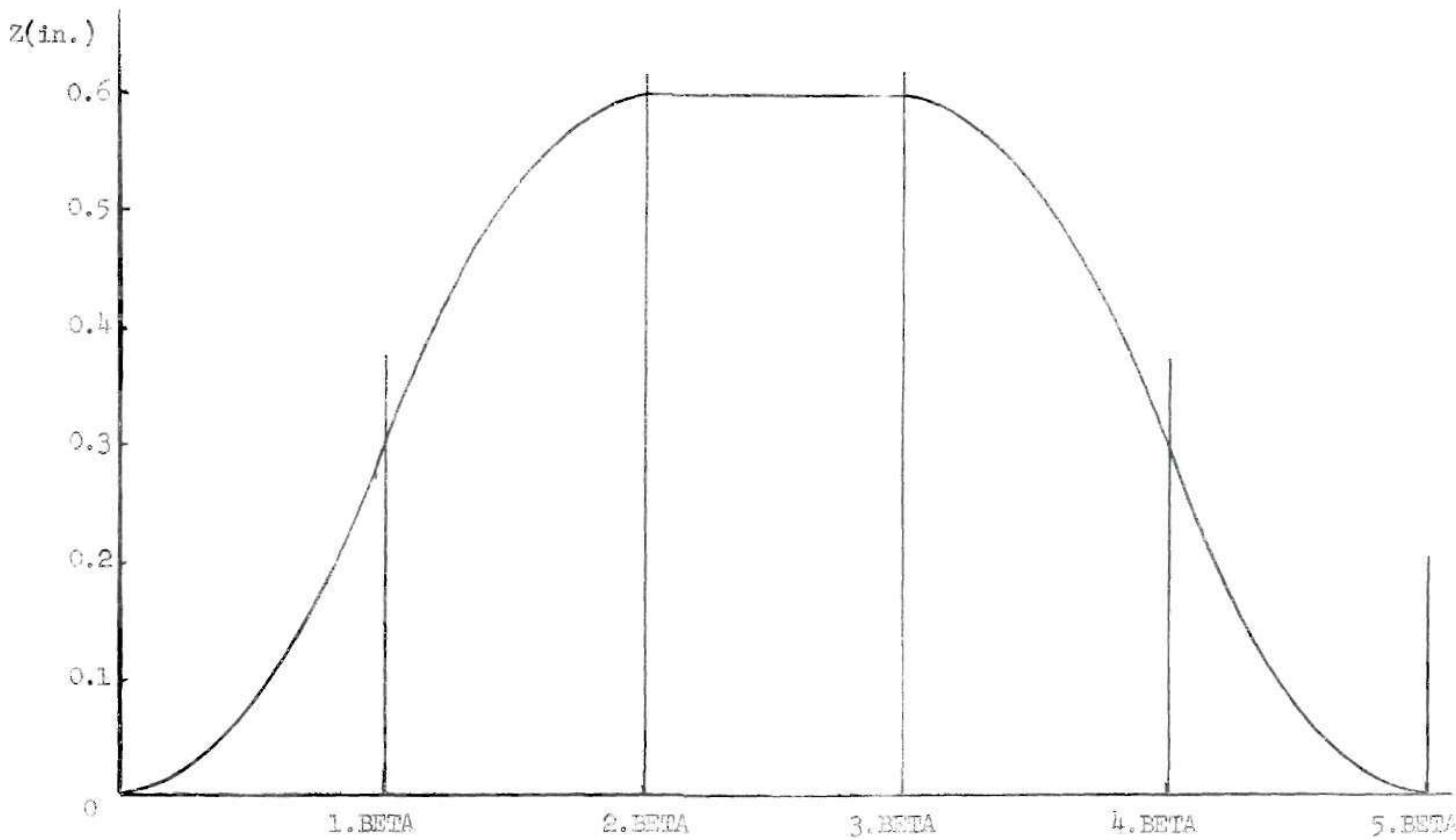


Figure 12. Cam Lift for Parabolic Cam Curve, Rise - Dwell - Return Motion.

```
EITHER IF T GTR (5.BETA)  $
    Z=0.0  $
 OR IF T GTR (4.BETA)  $
    Z=(2.H)(((5.BETA)-T)/(2.BETA))*2  $
 OR IF T GTR (3.BETA)  $
    Z=(H)(1.0-2.0(1.0-((5.BETA)-T)/(2.BETA))*2)  $
 OR IF T GTR (2.BETA)  $
    Z=H  $
 OR IF T GTR (BETA)  $
    Z=(H)(1.0-2.0(1.0-(T/(2.BETA)))*2)  $
 OTHERWISE  $
    Z=(2.H)((T/(2.BETA))*2)  $
```

APPENDIX B

COMPUTER PROGRAMS

```
0200                    BAC-220 STANDARD VERSION    2/1/62
0200        COMMENT  BERNARD KNIGHT THESIS.  RUNGE KUTTA SOLUTION FOR
0200                 D2X+5D1X+7X=0  WITH  X(0)=0 AND D1X(0)=10SQRT3/2            $
0200                 INTEGER K, N, I, Q                                          $
0200                 ARRAY D(25,25)                                              $
0200                 ARRAY X(2)=(0.0,8.660255)                                   $
0200                 N = 2                                                       $
0202                 DT=0.01                                                     $
0204                 M=0.60                                                      $
0206                 WRITE ($$TITLE1)                                            $
0210                 T=0.0                                                       $
0211        1..      FOR K = (1,1,4)                                             $
0222                 BEGIN
0223                 D(1,K)=(DT)(X(2))                                           $
0227                 D(2,K)=(DT)((-5.0)(X(2))-(7.0)(X(1)))                       $
0239                 IF MOD(K,2) EQL 1                                           $
0245                 T = T + (0.5)DT                                             $
0249                 EITHER IF K EQL 4                                           $
```

```
0249            GO 2                                                         $
0255            OTHERWISE                                                    $
0255            EITHER IF K EQL 3                                            $
0255            P = 1.0                                                      $
0263            OTHERWISE                                                    $
0263            P = 0.5                                                      $
0266            FOR I = (1,1,N)                                              $
0277            X(I)=X(I) + P.D(I,K)                                         $
0290            IF K NEQ 1                                                   $
0294    BEGIN   FOR I = (1,1,N)                                              $
0305            X(I)=X(I)-(0.5)(D(I,K-1))     END     END                    $
0319    2..     FOR I = (1,1,N)                                              $
0330            X(I)=X(I)+(D(I,1)+2.D(I,2)+2.D(I,3)+D(I,4))/6-D(I,3)         $
0379            X1C = 10.0000(EXP((-2.50000)T))(SIN(((SQRT(3.0000))/2.000)T)) $
0395            WRITE ($$ANS,FMT)                                            $
0403    OUTPUT  ANS(T,X(1),X1C)                                              $
```

```
0416      FORMAT   TITLE1(B7,*T                    X1                  X1C*,W2)  $
0429      FORMAT   FMT(B1,S7.3,2S22.6,W0)                                        $
0435               IF T LSS M                                                    $
0435               GO 1                                                          $
0441               FINISH                                                        $
COMPILED PROGRAM ENDS AT 0442
PROGRAM VARIABLES BEGIN AT 3863
```

T	X1	X1C
.010	.084463	.084463
.020	.164749	.164749
.030	.241007	.241007
.040	.313382	.313382
.050	.382012	.382012
.060	.447035	.447035
.070	.508582	.508582
.080	.566779	.566779
.090	.621751	.621751
.100	.673618	.673618
.110	.722495	.722495
.120	.768495	.768495
.130	.811727	.811727
.140	.852297	.852297
.150	.890306	.890306
.160	.925853	.925853
.170	.959034	.959034
.180	.989943	.989943
.190	1.01866	1.01866
.200	1.04529	1.04529
.210	1.06991	1.06991
.220	1.09259	1.09259
.230	1.11343	1.11343
.240	1.13248	1.13248
.250	1.14984	1.14984
.260	1.16556	1.16556
.270	1.17972	1.17972
.280	1.19238	1.19238
.290	1.20362	1.20362
.300	1.21348	1.21348
.310	1.22203	1.22203
.320	1.22933	1.22933
.330	1.23544	1.23544
.340	1.24041	1.24041
.350	1.24428	1.24428
.360	1.24712	1.24712
.370	1.24897	1.24897
.380	1.24987	1.24987
.390	1.24988	1.24988
.400	1.24903	1.24903
.410	1.24737	1.24737
.420	1.24494	1.24494
.430	1.24178	1.24178
.440	1.23793	1.23793
.450	1.23342	1.23342
.460	1.22829	1.22829
.470	1.22256	1.22256
.480	1.21629	1.21629
.490	1.20949	1.20949
.500	1.20219	1.20219
.510	1.19443	1.19443
.520	1.18623	1.18623
.530	1.17762	1.17762
.540	1.16863	1.16863
.550	1.15928	1.15928
.560	1.14960	1.14959
.570	1.13960	1.13960
.580	1.12931	1.12931
.590	1.11875	1.11875
.600	1.10794	1.10794

```
0200              BAC-220 STANDARD VERSION   2/1/62
0200      COMMENT  BERNARD KNIGHT THESIS.  RUNGE KUTTA SOLUTION FOR
0200               BARKANS SYSTEM OF ONE DEGREE OF FREEDOM FORCED BY
0200               FORCED BY PARABOLIC CAM CURVE                          $
0200               INTEGER K, N, I, Q                                     $
0200               ARRAY D(25,25)                                         $
0200               ARRAY X(2) = (0.0,0.0)                                 $
0200               N = 2                                                  $
0202               H=0.6        $
0204               BETA = 0.00215    $
0206               DT=0.000005                                            $
0208               M = 5.BETA                                             $
0211               WRITE ($$TITLE1)                                       $
0215               Q = 0                                                  $
0216               T=0.0                                                  $
0217      1..      FOR K = (1,1,4)                                        $
0228               BEGIN
0231               EITHER IF T GTR (5.BETA) $
```

```
0231                Z=0.0 $
0236            OR IF T GTR (4.BETA) $
0245                Z=(2.H)(((5.BETA)-T)/(2.BETA))*2  $
0260            OR IF T GTR (3.BETA) $
0267                Z=(H)(1.0-2.0(1.0-((5.BETA)-T)/(2.BETA))*2)    $
0286            OR IF T GTR (2.BETA) $
0286                Z=H  $
0295            OR IF T GTR (BETA) $
0301                Z=(H)(1.0-2.0(1.0-(T/(2.BETA)))*2)   $
0316            OTHERWISE  $
0319                Z=(2.H)((T/(2.BETA))*2)   $
0330        D(1,K) = (DT)(X(2))                                          $
0334        D(2,K) = (DT)((32104.Z-(3.24**4)X(1))/(3.286**-3))           $
0348        IF MOD(K,2) EQL 1                                            $
0354        T = T + (0.5)DT                                              $
0358        EITHER IF K EQL 4                                            $
```

```
0358            GO 2                                                  $
0364            OTHERWISE                                             $
0364            EITHER IF K EQL 3                                     $
0364            P = 1.0                                               $
0372            OTHERWISE                                             $
0372            P = 0.5                                               $
0375            FOR I = (1,1,N)                                       $
0386            X(I)=X(I) + P.D(I,K)                                  $
0399            IF K NEQ 1                                            $
0403    BEGIN   FOR I = (1,1,N)                                       $
0414            X(I)=X(I)-(0.5)(D(I,K-1))     END     END             $
0428    2..     FOR I = (1,1,N)                                       $
0439            X(I)=X(I)+(D(I,1)+2.D(I,2)+2.D(I,3)+D(I,4))/6-D(I,3)  $
0488            ACN = (32104.Z-(3.24**4)X(1))/(3.286**-3)             $
0497            DIFF = Z- X(1)                                        $
0500            Q = Q + 1                                             $
0503            IF MOD(Q,20) EQL 0                                    $
0513            WRITE ($$ANS,FMT)                                     $
```

```
0517        OUTPUT   ANS(T,X(1),X(2),ACN,Z,DIFF)                                        $
0539        FORMAT   TITLE1(B6,*T            X            VEL            ACN       Z
0539            DIFF*,W2)                                                               $
0556        FORMAT   FMT(B1,S8.6,5S12.6,W0)                                             $
0562                 IF T LSS M                                                         $
0562                 GO 1                                                               $
0568                 FINISH                                                             $
COMPILED PROGRAM ENDS AT 0569
PROGRAM VARIABLES BEGIN AT 4030
```

```
0200                 BAC-220 STANDARD VERSION   2/1/62
0200      COMMENT  BERNARD KNIGHT THESIS   RUNGE KUTTA SOLUTION FOR
0200               BARKANS SYSTEM C MODIFIED                             $
0200              INTEGER K, N, I, Q                                     $
0200              ARRAY D(25,25)                                         $
0200              ARRAY X(8) = (0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0)         $
0200              N = 8                                                  $
0202              H=0.6  $
0204              BETA = 0.00215    $
0206              DT=0.000005    $
0208              M = 5.BETA                                             $
0211              WRITE ($$TITLE1)                                       $
0215              Q = 0                                                  $
0216              T=0.0                                                  $
0217      1..     FOR K = (1,1,4)                                        $
0228              BEGIN
0231              EITHER IF T GTR (5.BETA) $
0231                 BEGIN
```

```
0231                Z=0.0 $
0236                 ZDD=0.0   $
0237             END   $
0237             OR IF T GTR (4.BETA) $
0237             BEGIN
0246                Z=(2.H)(((5.BETA)-T)/(2.BETA))*2   $
0261                 ZDD=H/((BETA)*2)   $
0268             END   $
0268             OR IF T GTR (3.BETA) $
0268             BEGIN
0275                Z=(H)(1.0-2.0(1.0-((5.BETA)-T)/(2.BETA))*2)    $
0294                 ZDD= -H/((BETA)*2)   $
0301             END   $
0301             OR IF T GTR (2.BETA) $
0301             BEGIN
0301                Z=H   $
```

```
0310                 ZDD=0.0  $
0311              END  $
0311              OR IF T GTR (BETA) $
0311              BEGIN
0317                Z=(H)(1.0-2.0(1.0-(T/(2.BETA)))*2)  $
0332                 ZDD= -H/((BETA)*2)  $
0339              END  $
0339              OTHERWISE  $
0339              BEGIN
0342                Z=(2.H)((T/(2.BETA))*2)  $
0353                 ZDD=H/((BETA)*2)  $
0360              END  $
0360          D(1,K) = (DT)(X(2))                                              $
0364          D(2,K) = (DT)(((-296.0)X(1) + (X(3)-X(1))(30**4))/1.333**-3)     $
0379          D(3,K) = (DT)(X(4)  )                                            $
0386          D(4,K) = (DT)((-(X(3)-X(1))(30**4) + (X(5)-X(3))(7.15**4))/
0397                    (1.075**-3))                                           $
0402          D(5,K) =(DT)(X(6))                                               $
```

```
0409          D(6,K) = (DT)((-(X(5)-X(3))(7.15**4) +  (X(7)-X(5))(6.90**4))/
0420                     (0.486**-3))                                          $
0425          D(7,K) = (DT)(X(8))                                              $
0432          D(8,K) = (DT)((-(X(7)-X(5))(6.90**4) + (Z-X(7))(11.8**4))/
0443                     (0.392**-3))                                          $
0448          IF MOD(K,2) EQL 1                                                $
0454          T = T + (0.5)DT                                                  $
0458          EITHER IF K EQL 4                                                $
0458          GO 2                                                             $
0464          OTHERWISE                                                        $
0464          EITHER IF K EQL 3                                                $
0464          P = 1.0                                                          $
0472          OTHERWISE                                                        $
0472          P = 0.5                                                          $
0475          FOR I = (1,1,N)                                                  $
0486          X(I)=X(I) + P.D(I,K)                                             $
```

```
0499              IF K NEQ 1                                                      $
0503     BEGIN    FOR I = (1,1,N)                                                 $
0514              X(I)=X(I)-(0.5)(D(I,K-1))     END    END                        $
0528     2..      FOR I = (1,1,N)                                                 $
0539              X(I)=X(I)+(D(I,1)+2.D(I,2)+2.D(I,3)+D(I,4))/6-D(I,3)            $
0588              ACN = X(1)(-296.0)/(1.333**-3) + (X(3)-X(1))(30**4)/
0596                        (1.333**-3)                                           $
0600              CAMFORCE = (11.8**4)(Z-X(7)) + (0.177**-3)ZDD                   $
0608              Q = Q + 1                                                       $
0611              IF MOD(Q,20) EQL 0                                              $
0621              WRITE ($$ANS,FMT)                                               $
0625     OUTPUT   ANS(T,X(1),X(2),ACN,CAMFORCE,Z,X(7))                            $
0650     FORMAT   TITLE1(B6,*T          X          VEL          ACN        CAMFORCE
0650              Z           X4*,W2)                                             $
0669     FORMAT   FMT(B1,S8.6,6S12.6,W0)                                          $
0675              IF T LSS M                                                      $
0675              GO 1                                                            $
0681              FINISH                                                          $
COMPILED PROGRAM ENDS AT 0682
PROGRAM VARIABLES BEGIN AT 4009
```

T	X	VEL	ACN	CAMFORCE	Z	X4
.000100	.000000	.000202	17.7377	83.1288	.000648	.000139
.000200	.000001	.071832	2879.11	158.921	.002595	.001443
.000300	.000054	1.48905	33604.9	225.828	.005840	.004121
.000400	.000484	8.35908	106304.	323.846	.0103=3	.007834
.000500	.001944	21.4844	144888.	431.184	.016224	.012765
.000600	.004825	36.2853	156024.	592.164	.023363	.018540
.000700	.009314	54.4302	210944.	714.153	.031800	.025943
.000800	.015887	77.6088	245733.	752.841	.041535	.035350
.000900	.024915	103.426	274192.	806.639	.052568	.045927
.001000	.036671	131.828	280899.	844.336	.064899	.057939
.001100	.051172	157.254	228003.	839.982	.078528	.071605
.001200	.068022	179.956	238343.	776.171	.093455	.087072
.001300	.087234	204.099	226811.	651.322	.109680	.104355
.001400	.108650	222.953	153231.	558.204	.127203	.122668
.001500	.131650	236.566	121556.	479.346	.146024	.142157
.001600	.155824	245.802	57054.7	358.617	.166143	.163299
.001700	.180612	249.627	36728.1	233.399	.187560	.185777
.001800	.205798	254.228	42248.8	154.723	.210275	.209159
.001900	.231319	254.899	-30750.5	142.573	.234288	.233275
.002000	.256618	251.259	-22525.1	161.396	.259599	.258426
.002100	.281710	251.142	10198.6	178.037	.286208	.284894
.002200	.306877	252.352	22040.8	140.758	.313791	.312403
.002300	.332318	257.504	79526.3	93.6058	.340400	.339412
.002400	.358475	265.243	57302.4	125.810	.365711	.364450
.002500	.385169	267.496	-9561.23	67.2706	.389724	.388959
.002600	.411791	264.253	-51301.7	-28.2385	.412439	.412483
.002700	.437922	258.085	-69151.5	-187.969	.433856	.435254
.002800	.463339	249.508	-114657.	-400.945	.453975	.457178
.002900	.487559	233.302	-208044.	-464.713	.472796	.476539
.003000	.509749	209.707	-256636.	-573.828	.490319	.494987
.003100	.529371	182.064	-295418.	-724.453	.506544	.512488
.003200	.546091	152.623	-280289.	-780.592	.521471	.527891
.003300	.560000	125.604	-279275.	-802.985	.535100	.541710
.003400	.571041	94.0933	-340930.	-699.684	.547431	.553165
.003500	.578822	62.9408	-260298.	-605.651	.558464	.563401
.003600	.583967	40.9687	-200554.	-562.876	.568199	.572774
.003700	.587078	21.7472	-167475.	-393.769	.576636	.579778
.003800	.588581	9.83092	-82342.2	-201.107	.583775	.585284
.003900	.589136	.788919	-104469.	-34.5362	.589616	.589714
.004000	.588780	-6.15465	-2333.84	73.6615	.594159	.593340
.004100	.588387	-.206074	83045.1	118.141	.597404	.596208
.004200	.588661	4.34894	11920.6	210.281	.599351	.597374
.004300	.589192	7.15098	57151.5	257.859	.600000	.597620
.004400	.590197	12.2368	18047.7	272.577	.600000	.597690
.004500	.591399	11.3978	-4454.77	213.072	.600000	.598194
.004600	.592645	14.5278	48973.7	154.519	.600000	.598690
.004700	.594219	15.3100	-42300.8	157.321	.600000	.598666
.004800	.595489	10.3549	-30128.1	109.799	.600000	.599069
.004900	.596446	8.93854	-22090.1	85.4650	.600000	.599275
.005000	.597128	3.99505	-60611.4	49.5883	.600000	.599579
.005100	.597305	.418610	-13773.7	27.4220	.600000	.599767
.005200	.597232	-2.81113	-64212.8	90.5744	.600000	.599232
.005300	.596587	-9.82988	-50496.5	76.2516	.600000	.599353
.005400	.595472	-11.6247	-1949.54	89.6457	.600000	.599240
.005500	.594258	-13.0811	-23704.0	140.128	.600000	.598812
.005600	.592877	-13.9001	10883.1	160.079	.600000	.598643
.005700	.591547	-12.9427	-1107.40	248.548	.600000	.597893
.005800	.590247	-12.6573	23203.9	248.305	.600000	.597895
.005900	.589193	-7.78205	59636.0	245.186	.600000	.597922

.006000	.588666	-3.26672	32557.2	313.023	.600000	.597347
.006100	.588518	.625905	47909.5	295.112	.600000	.597499
.006200	.588804	4.70970	27552.7	305.603	.600000	.597410
.006300	.589408	7.66928	44505.0	270.026	.600000	.597711
.006400	.590442	13.0379	43861.1	208.789	.600000	.598230
.006500	.591851	14.1038	-14000.7	195.496	.599837	.597986
.006600	.593211	13.5395	7473.60	48.4307	.598539	.597934
.006700	.594570	12.8460	-38062.2	-68.6417	.595943	.596330
.006800	.595551	5.89395	-90434.4	-177.895	.592049	.593362
.006900	.595624	-5.40181	-150108.	-321.533	.586857	.589387
.007000	.594161	-25.3153	-232548.	-423.070	.580367	.583758
.007100	.590492	-47.5087	-205921.	-608.635	.572579	.577543
.007200	.584665	-70.1343	-267555.	-699.909	.563493	.569230
.007300	.576188	-100.059	-305987.	-689.968	.553109	.558762
.007400	.564716	-128.913	-282465.	-717.349	.541427	.547312
.007500	.550355	-158.794	-304579.	-678.651	.528447	.534004
.007600	.533085	-184.796	-203247.	-635.452	.514169	.519360
.007700	.513677	-203.337	-198656.	-512.547	.498593	.502742
.007800	.492277	-224.686	-196180.	-325.574	.481719	.484284
.007900	.469049	-237.642	-70734.2	-230.849	.463547	.465309
.008000	.444972	-243.962	-67510.1	-119.474	.444077	.444895
.008100	.420314	-247.840	10640.6	-11.7150	.423309	.423214
.008200	.395702	-244.068	29123.0	88.0256	.401243	.400303
.008300	.371303	-244.885	-21563.1	148.064	.377879	.376430
.008400	.346857	-242.250	76363.3	80.0866	.353217	.352344
.008500	.323002	-235.907	17638.5	22.2477	.327278	.326874
.008600	.299345	-238.101	-33297.9	-49.8667	.300000	.300227
.008700	.275402	-240.862	-44606.2	6.62568	.272742	.272880
.008800	.250899	-251.068	-148281.	11.7457	.246782	.246877
.008900	.225116	-262.980	-57899.1	-63.8734	.222120	.222856
.009000	.198726	-263.577	15660.0	23.1740	.198756	.198754
.009100	.172399	-263.371	-4382.13	150.707	.176690	.175607
.009200	.146114	-261.342	52821.5	369.929	.155922	.152981
.009300	.120348	-252.928	117165.	565.822	.136452	.131851
.009400	.095802	-236.179	224366.	600.784	.118280	.113383
.009500	.073462	-209.480	292193.	803.628	.101406	.094790
.009600	.053974	-180.430	284749.	989.466	.085830	.077639
.009700	.037349	-151.983	290823.	1030.04	.071552	.063017
.009800	.023679	-120.447	347826.	1042.88	.058572	.049928
.009900	.013464	-83.4256	370670.	911.417	.046890	.039361
.010000	.006827	-51.1475	265662.	875.184	.036506	.029284
.010100	.002911	-27.9773	215052.	828.561	.027420	.020593
.010200	.001146	-8.07063	169430.	560.194	.019632	.015079
.010300	.001058	5.35073	114615.	357.836	.013142	.010304
.010400	.002180	17.0087	103684.	192.814	.007950	.006510
.010500	.004167	19.8288	-63153.4	81.3927	.004056	.003561
.010600	.005666	9.73382	-88829.0	15.3064	.001460	.001525
.010700	.006375	5.75670	-17134.4	-155.962	.000162	.001678

APPENDIX C

CALCULATION OF EFFECTIVE MASS OF THE VALVE SPRING

The valve spring is not simply a reciprocating link in the system. It is moving at the valve speed at one end while remaining stationary at the other end. Therefore not all the mass of the valve spring contributes to the mass of the system. The effective mass can be calculated by a method first proposed by Rayleigh as described by Barkan.[6]

Consider the system given in Figure 13, consisting of a mass M and a spring of length p having a mass per unit length of m. As the mass oscillates about its equilibrium position $(x << p)$, a displacement x of the mass M produces a displacement z of an element dy situated at a point y on the spring. Then $z = \frac{y}{p} x$, provided the frequency of vibration is very low compared to the natural frequency of the spring, and $\frac{dz}{dt} = \frac{y}{p} \frac{ds}{dt}$.

The contribution of the element dy to the kinetic energy, dT, of the system is

$$dT = \frac{1}{2} [m\ dy] [\frac{y}{p} \mathring{x}]^2, \quad \text{where} \quad \mathring{x} = \frac{dx}{dt} .$$

Thus the kinetic energy of the entire spring is

$$Ts = \frac{1}{2} m \frac{\mathring{x}^2}{p^2} \int_o^p y^2\ dy$$

$$= \frac{1}{2} \mathring{x}^2 \frac{mp}{3} .$$

If m_s is the total mass of the spring, mp,

$$T_s = \frac{1}{2} [\frac{m_s}{3} \dot{x}^2] .$$

The kinetic energy of the mass M is

$$T_M = \frac{1}{2} M \dot{x}^2 ;$$

hence the total kinetic energy is given by $\frac{1}{2} [M + \frac{m_s}{3}] \dot{x}^2$. Therefore one third of the mass of the valve spring is used in calculating the effective mass of the system.

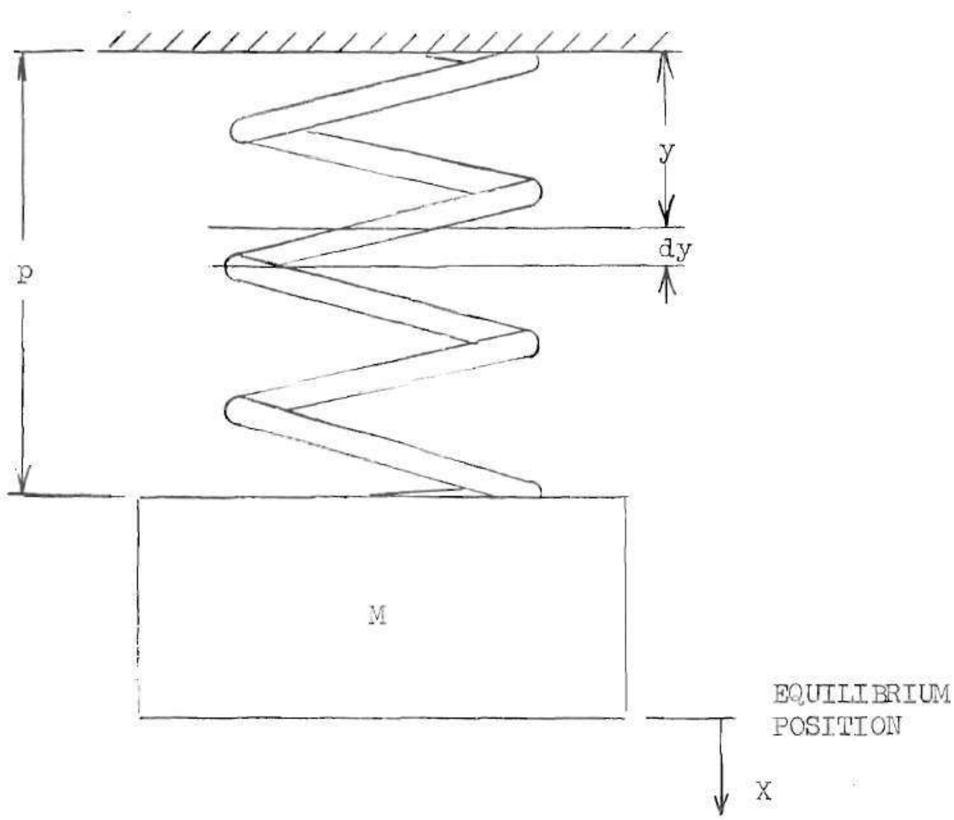


Figure 13. Valve Spring with Lumped Mass.

APPENDIX D

REACTION FORCE AT THE CAM - A CONTINUUM ANALYSIS

The overhead valve mechanism as shown in Figure 1 can be more accurately represented as a continuum rather than as a lumped parameter system as shown in Figure 14.

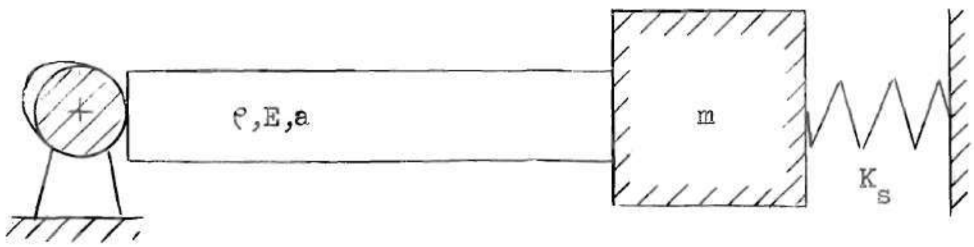


Figure 14. Continuum Representation of Overhead Valve Mechanism.

The first step in this analysis is to develop the transfer matrix[15,16] of the continuum itself as shown in Figure 15, where ρ is the density, a is the cross sectional area, E is Young's Modulus of the continuum material, and λ is the terminal mobility (to be defined and discussed later). The governing differential equation of the continuum is the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} , \qquad (67)$$

where $c = \sqrt{\frac{E}{\rho}}$. Let $s = \frac{d}{dt}$, so that equation (67) becomes

$$\frac{d^2y}{dx^2} - \left(\frac{s}{c}\right)^2 y = 0 \; . \tag{68}$$

The general operational solution of equation (68) is

$$y = A \cosh (s/c)x + B \sinh (s/c)x \; . \tag{69}$$

The boundary conditions are $y_o = y_1$; $-Ea(\partial y/\partial x)_o = f_1$ at $x = 0$ and $y_\ell = y_2$; $-E\,a\,(\partial y/\partial x)_\ell = f_2$, from which the constants A and B in the general solution are calculated. Then the solution becomes

$$y = y_1 \cosh (s/c)x - \frac{1}{a\sqrt{E\rho}} \frac{1}{s} f_1 \sinh (s/c)x \; . \tag{70}$$

By using the remaining boundary conditions, the state vector[16]

$$\begin{bmatrix} y_1 \\ f_1 \end{bmatrix}$$

that is, the displacement and force conditions, at the cam end can be determined as

$$y_1 = \left(\cosh \frac{s}{c} \ell\right) y_2 + \frac{\left(\sinh \frac{s}{c} \ell\right)}{a\sqrt{E\rho}\;\; s} f_2 \tag{71}$$

$$f_1 = a\sqrt{E\rho}\; s \sinh\left(\frac{s}{c}\ell\right) y_2 + \cosh\left(\frac{s}{c}\ell\right) f_2 \, , \tag{72}$$

which can be written in matrix notation as follows:

$$\begin{Bmatrix} y_1 \\ f_1 \end{Bmatrix} = \begin{bmatrix} \cosh\left(\frac{s}{c}\ell\right) & \dfrac{\sinh\left(\frac{s}{c}\ell\right)}{a\sqrt{E\rho}\; s} \\ a\sqrt{E\rho}\; s \sinh\left(\frac{s}{c}\ell\right) & \cosh\left(\frac{s}{c}\ell\right) \end{bmatrix} \begin{Bmatrix} y_2 \\ f_2 \end{Bmatrix} . \qquad (73)$$

At this point define:

$$\text{mobility}^{15} = \frac{\text{relative displacement across element}}{\text{force across element}} = \lambda \ .$$

For example, the mobility of a spring is $\frac{1}{k}$, where k is the spring constant. Now the overhead valve system representation in Figure 14 can be shown as being represented by Figure 15. The mobility of the mass-

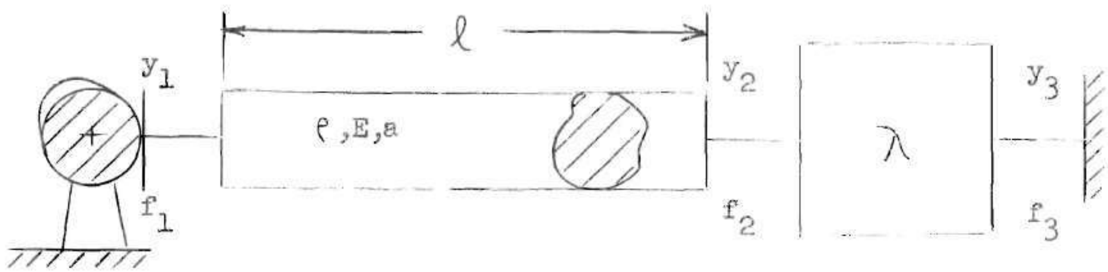


Figure 15. Continuum with Terminal Mobility.

spring terminal combination in Figure 14 can be determined as depicted in Figure 16.

At this point let the transfer matrix shown in equation (73) be represented by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} ,$$

where $A = \cosh\left(\frac{s}{c}\ell\right)$, $B = \dfrac{\sinh\left(\frac{s}{c}\ell\right)}{a\sqrt{E\rho}\cdot s}$, $C = a\sqrt{E\rho}\cdot s\cdot\sinh\left(\frac{s}{c}\ell\right)$, and $D = \cosh\left(\frac{s}{c}\ell\right)$.

Further let y_1 be given by $Z(t)$, the input function or cam curve as a function of time. Also notice that, since y_3, the displacement of the wall, is zero, $y_2 = \lambda f_2$ so that the matrix equation (73) becomes

$$\begin{Bmatrix} Z(t) \\ f_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} \lambda f_2 \\ f_2 \end{Bmatrix}, \tag{74}$$

where $\lambda = \dfrac{1}{ms^2 + k}$ for the spring-mass combination shown in Figure 16. For a unit step input it can be shown that $Z(t) = \frac{1}{s}\delta$, where δ is the Dirac delta[2] or unit impulse function having an "infinite" ordinate, a "zero" abscissa, and an area of one under its curve. With a unit step input equation (74) becomes

$$f_1 = \frac{C\lambda + D}{A\lambda + B}\cdot Z(t)$$

$$\frac{a\sqrt{E\rho}\ s\ \sinh\left(\frac{s}{c}\ell\right)\cdot\dfrac{1}{ms^2 + k} + \cosh\left(\frac{s}{c}\ell\right)}{\cosh\left(\frac{s}{c}\ell\right)\cdot\dfrac{1}{ms^2 + k} + \dfrac{1}{a\sqrt{E\rho}\ s}\sinh\left(\frac{s}{c}\ell\right)}\cdot\frac{1}{s}\delta \tag{75}$$

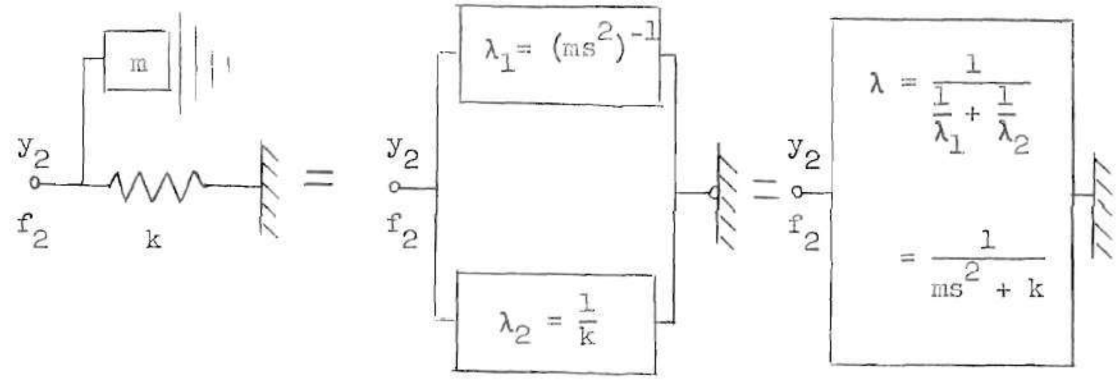


Figure 16. Mobility of Mass-Spring Combination

Now, let $p = -j(\frac{s}{c}\ell)$ so that $s = j(\frac{c}{\ell})p$ and $s^2 = -(\frac{c}{\ell})^2 p^2$, where $c = \sqrt{E/\rho}$. Further define the mass ratio coefficient $\eta = \frac{\ell\rho a}{m}$ and the stiffness ratio $\mu = \frac{Ea/\ell}{k}$ Making the above substitutions, equation (75) becomes

$$f_1 = \frac{a\sqrt{E\rho}}{j} \cdot \frac{\eta \frac{p \sin p}{p^2 - \eta/\mu} + \cos p}{-\eta \frac{p \cos p}{p^2 - \frac{\eta}{\mu}} + \sin p} \cdot \delta . \tag{76}$$

Equation (76) is the operational form for the reaction force at the cam due to a step input. Notice that it has the form

$$f_1 = \frac{P(p)}{Q(p)} \cdot \delta .$$

Now, from the Heaviside expansion theorem[3]

$$f_1 = \sum_{i=1,2,3,\ldots} \frac{P(p_i)\, e^{j(\frac{c}{\ell})p_i t}}{\left[\frac{dQ(p)}{dp} \cdot \frac{dp}{ds}\right]_{p=p_i}} , \tag{77}$$

where p_i are the successive roots or eigenvalues of $Q(p) = 0$. The terms in the denominator of equation (77) are written as follows:

$$\frac{dp}{ds} = -j\left(\frac{\ell}{c}\right) = -j\,\ell\sqrt{\frac{\rho}{E}} \qquad \text{(a)}$$

$$Q(p) = -\eta\, p \cos p + \left(p^2 - \frac{\eta}{\mu}\right) \sin p$$

$$\frac{d\,Q(p)}{dp} = (2+\eta)p \sin p + \left[p^2 - \frac{\eta(1+\mu)}{\mu}\right] \cos p \qquad \text{(b)}$$

Substituting equations (76), (a), and (b) into equation (77) the reaction force at the cam becomes

$$f_1 = \left(\frac{aE}{\ell}\right) \sum_{i=1,2,3\ldots} \frac{\eta p_i \sin p_i + \left(p_i^2 - \frac{\eta}{\mu}\right) \cos p_i}{(2+\eta)p_i \sin p_i + \left[p_i^2 - \frac{\eta(1+\mu)}{\mu}\right] \cos p_i}\, e^{j\left(\frac{c}{\ell}\right)p_i t} \qquad (78)$$

where p_i are the successive roots of $Q(p_i) = 0$. It can be shown that when p_i is a root of $Q(p_i) = 0$, then $-p_i$ is also a root. It is also seen that $p_0 = 0$; consequently the first term in the series is

$$\left(\frac{aE}{\ell}\right) \frac{\eta\cdot 0\cdot \sin(0) + \left(0 - \frac{\eta}{\mu}\right) \cos(0)}{(2+\eta)\cdot 0 \cdot \sin(0) + \left(0 - \frac{\eta(1+\mu)}{\mu}\right) \cos(0)}\, e^{(0)}$$

$$= \frac{aE}{\ell} \cdot \frac{1}{1+\mu}$$

$$= \frac{1}{\frac{1}{aE/\ell} + \frac{1}{k}}$$

This first term is the equivalent stiffness of the continuum and the valve spring.

A pair of the $\pm\, i^{th}$ terms in the series (78) will be η

$$\frac{\eta p_i \sin p_i + (p_i^2 - \frac{\eta}{\mu}) \cos p_k}{(2+\eta) p_i \sin p_i + \left\{p_i^2 - \frac{\eta(1+\mu)}{\mu}\right\} \cos p_i} e^{j(\frac{c}{\ell}) p_i t}$$

$$+ \frac{\eta(-p_i) \sin(-p_i) + (p_i^2 - \frac{\eta}{\mu}) \cos(-p_i)}{(2+\eta)(-p_i) \sin(-p_i) + \left\{p_i^2 - \frac{\eta(1+\mu)}{\mu}\right\} \cos(-p_i)} e^{-j(\frac{c}{\ell}) p_i t}$$

$$= 2\left\{\frac{\eta p_i \sin p_i + (p_i^2 - \frac{\eta}{\mu}) \cos p_i}{(2+\eta) p_i \sin p_i + \left[p_i^2 - \frac{\eta(1+\mu)}{\mu}\right] \cos p_i}\right\} \cos\left(\frac{c}{\ell}\right) p_i t$$

Therefore the step response of the system, using the notation

$$\left\{\text{coef.}\right\} = \frac{\eta p_i \tan p_i + (p_i^2 - \frac{\eta}{\mu})}{(2+\eta) p_i \tan p_i + \left\{p_i^2 - \frac{\eta(1+\mu)}{\mu}\right\}},$$

becomes

$$f_1 = \frac{1}{\frac{1}{aE/\ell} + \frac{1}{k}} + 2\left(\frac{aE}{\ell}\right) \sum_{i=1,2,3,\ldots} \left\{\text{coef.}\right\} \cos p_i \left(\frac{c}{\ell}\right) t \tag{79}$$

where p_i is the positive root of

$$-\eta p_i \cos p_i + (p_i^2 - \frac{\eta}{\mu}) \sin p_i = 0 ,$$

or

$$\tan p_i = \frac{\eta p_i}{(p_i^2 - \frac{\eta}{\mu})} . \tag{80}$$

Substituting for tan p_i from equation (80),

$$\left\{ \text{coef.} \right\} = 1 - \frac{\eta (p_i^2 + \frac{\eta}{\mu})}{p_i^4 + \eta(1 + \eta - \frac{2}{\mu})\, p_i^2 + (1 + \mu)(\frac{\eta}{\mu})^2} . \tag{81}$$

So that the response to a unit step input becomes

$$f_1(t)_{step} = \frac{1}{\frac{1}{aE/\ell} + \frac{1}{k}} + 2\left(\frac{aE}{\ell}\right) \sum_{i=1,2,3,\ldots} \left[1 - \frac{\eta(p_i^2 + \frac{\eta}{\mu})}{p_i^4 + \eta(1+\eta - \frac{2}{\mu}) p_i^2 + (1+\mu)(\frac{\eta}{\mu})^2} \right] \cdot \cos p_i \left(\frac{c}{\ell}\right) t$$

$$= B(t) \tag{82}$$

When the unit step response to a system, $B(t)$, is given the response due to any arbitrary input $Z(t)$ is given by Duhamel's integral[2] in the form

$$f(t) = Z(0)B(t) + \int_0^t \frac{dZ(\tau)}{d\tau} B(t - \tau)\, d\tau . \tag{83}$$

Let equation (82) be written in the form

$$B(t) = K + \sum_{i=1,2,3,\ldots} M_i \cos \frac{p_i}{T} t . \tag{84}$$

For a polynomial cam curve

$$Z(t) = \sum_{n=o}^{n} A_n t^n , \quad \text{where} \quad A_o = 0 , \tag{85}$$

and

$$Z^{(h)}(t) = \sum_{n=h}^{n=m} \frac{\underline{|n}}{\underline{|n-h}} A_n t^n \, . \tag{86}$$

Equation (83) becomes

$$f(t) = \int_0^t \frac{dZ(\tau)}{d\tau} \left\{ K + \sum_{i=1,2,3,\ldots} M_i \cos \left(\frac{p_i}{T}\right)(t-\tau) \right\} d\tau$$

$$= K\Big[Z(t) - Z(0)\Big] + \sum_{i=1,2,3,\ldots} M_i \int_0^t \frac{dZ(\tau)}{d\tau} \cos\left(\frac{p_i}{T}\right)(t-\tau)d\tau, \tag{87}$$

where

$$K = \frac{1}{\frac{1}{\left(\frac{aE}{\ell}\right)} + \frac{1}{k}} \quad \text{and} \quad M_i = 2\left(\frac{aE}{\ell}\right)\left[1 - \frac{\eta\left\{p_i^2 + \frac{\eta}{\mu}\right\}}{p_i^4 + \eta\left(1+\eta-\frac{2}{\mu}\right)p_i^2 + (1+\mu)\left(\frac{\eta}{\mu}\right)^2}\right].$$

Using the relationship

$$\cos(t-\tau) = \cos t \cos \tau + \sin t \sin \tau$$

and integrating again by parts each of the two resulting integrals.

$$\int_0^t \frac{dZ(\tau)}{d\tau} \cos a_i(t-\tau)d\tau$$

$$= \cos a_i t \int_0^t \frac{dF(\tau)}{d\tau} \cos a_i\tau d\tau + \sin a_i t \int_0^t \frac{dF(\tau)}{d\tau} \sin a_i \tau d\tau$$

$$= - \sum_{n=1,2,3,\ldots} (-1)^n \frac{Z^{(2n)}(t)}{a^{2n}} +$$

$$\sqrt{\left\{\sum_{n=1,2,3,\ldots} (-1)^n \frac{Z^{(2n)}(0)}{a^{2n}}\right\}^2 + \left\{\sum_{n=1,2,3,\ldots} (-1)^n \frac{Z^{(2n-1)}(0)}{a^{2n-1}}\right\}^2} \cdot$$

$$\cos\left\{a_i t + \varphi\right\} \tag{88}$$

where

$$\varphi = \arctan \frac{\displaystyle\sum_{n=1,2,3,\ldots} (-1)^n \frac{Z^{(2n-1)}(0)}{a^{2n-1}}}{\displaystyle\sum_{n=1,2,3,\ldots} (-1)^n \frac{Z^{(2n)}(0)}{a^{2n}}}$$

For a parabolic cam curve in the first half of the rise $Z(t) = A_2 t^2$, and equation (87) after substituting equation (88) becomes

$$f(t) = \frac{A_2 t^2}{\frac{1}{aE/\nu} + \frac{1}{k}} +$$

$$4A_2(a^{\nu}\rho) \sum_{i=1,2,3,\ldots} \frac{1}{p_i^{\,2}} \left[1 - \frac{\eta\left\{p_i^{\,2} + (\frac{\eta}{\mu})\right\}}{p_i^{\,4} + \eta(1+\eta-\frac{2}{\mu})p_i^{\,2} + (1+\mu)(\frac{\eta}{\mu})^2}\right]\left\{1 - \cos p_i(\frac{c}{\ell})t\right\} \tag{89}$$

where p_i is given by roots of

$$\cot p_i = \frac{p_i}{\eta} - \frac{1}{\mu p_i} \cdot \tag{90}$$

The roots of equation (80) or (90) can be determined graphically as shown in Figure 17.

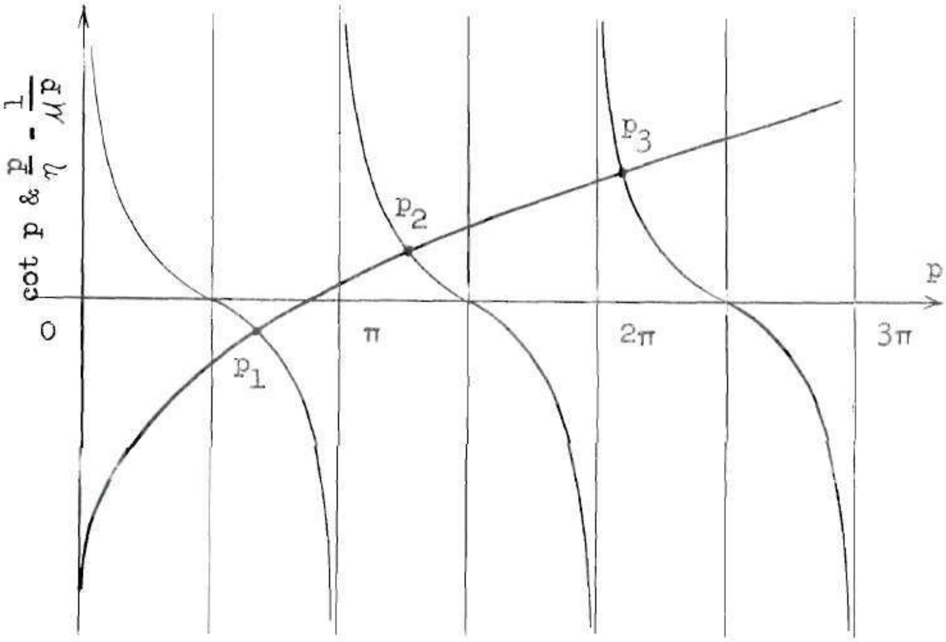


Figure 17. The roots of Equation (90).

LITERATURE CITED

1. E. H. Olmstead and E. S. Taylor, "Poppet Valve Dynamics," Journal of Aeronautical Sciences, Vol. 6, 1938-1939, pp. 370.

2. T. Von Kármán and M. Biot, Mathematical Methods in Engineering, McGraw-Hill, 1940, Chapter 10.

3. T. Nakada, Automatic Control and Servo-Mechanisms, published in Japan.

4. F. A. Willers, Practical Analysis, Chapter 6.

5. P. Barkan, "Calculation of High Speed Motion," Transactions SAE, Vol. 61, pp. 687 (1953).

6. P. Barkan, Calculation of High Speed Motion of Flexible Cam-Actuated Systems, Doctor of Philosophy dissertation, The Pennsylvania State College, Jan. 1953.

7. W. M. Dudley, "New Methods in Valve Cam Design," Transactions SAE, Vol. 2, pp. 19, Jan. 1948.

8. D. A. Stoddart, "Polydyne Cam Design," Machine Design, Vol. 25, pp. 121, Jan. 1953; pp. 146, Feb. 1953; pp. 149, Mar. 1953.

9. J. A. Hrones, "An Analysis of the Dynamic Forces in a Cam Driven System," Transactions ASME, Vol. 70, 1948, pp. 473.

10. R. Techo, A Problem Language Primer, Rich Electronic Computer Center, Georgia Institute of Technology.

11. D. B. Mitchell, "Tests on Dynamic Response of Cam-Follower Systems," Mechanical Engineering, Vol. 72, June, 1950.

12. T. Warming, "Vibrations in Valve Mechanisms," Proceedings of the 21st National Oil and Gas Power Conference, ASME, April 1949.

13. L. S. Jacobsen and R. S. Ayre, Engineering Vibrations, McGraw-Hill, 1958, Chapter 3.

14. Harold Rothbart, Cams, John Wiley and Sons, Inc.

15. T. Nakada, Matrix Analysis of Mechanical Systems, Bul. of the Tokyo Institute of Technology Series B 1956, Number 2; 621.53.

16. E. C. Pestel and F. A. Leckie, Matrix Methods in Elastomechanics, McGraw-Hill, 1963.

ADDITIONAL REFERENCES

1. Anderson, D. G., "Cam Dynamics," Product Engineering, vol. 24, page 170, October, 1953.

2. Churchill, R. V., Operational Mathematics, McGraw-Hill, 1958.

3. Churchill, R. V., Complex Variables and Applications, McGraw-Hill, 1960.

4. Crossley, F. R. E., Dynamics in Machines, Ronald Press, 1954.

5. Gutman, A. S., "Cam Dynamics," Machine Design, vol. 23.

6. Kaplan, W., Ordinary Differential Equations, Addison-Wesley, 1961.

7. Milne, W. E., Numerical Solution of Differential Equations, Chapter 5, John Wiley and Sons, 1953.

8. Nakada, T., Unpublished Work (Handwritten Notes).

9. Neklutin, C. N., "Cams for High Speeds," Product Engineering, July, 1934, page 250, August, 1934, page 296.

10. Neklutin, C. N., "Designing Cams for Controlled Inertia and Vibration," Machine Design, vol. 24, June 1952.

11. Neklutin, C. N., "Vibration Analysis of Cams," Machine Design, December, 1954.

12. Srinivasan, P., "Transient Response of Multi-Degree Freedom Systems to Unit Step," Journal of the Institution of Engineers (India), Vol. 44, 1964, No. 7, ME4.

13. Timoshenko and Young, Advanced Dynamics, McGraw-Hill, 1948.

14. Timoshenko, Vibration Problems in Engineering, Van Nostrand, 1955.